Simulations of Communication Systems via Integrated Variance Reduction Techniques

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Abstract—This paper deals with efficiency issues of estimating common error performance measures of communication systems via Monte Carlo simulations. Efficient simulation techniques using integrated variance reduction techniques (VRTs) are proposed and studied. The integrated techniques are constructed by combining a number of basic VRTs. The general idea is that by integrating several basic VRTs their respective complementary properties can be exploited simultaneously, leading to additional variance reductions of the estimates and, consequently, to substantial gains in terms of simulation time required to obtain them.

Based on the basic VRTs, two examples of the integrated techniques are presented and their performance is studied. Simulation results are provided to demonstrate the general approach of the integrated VRTs and to verify the concept. The results indicate that the simulation schemes based on the integrated VRTs can, indeed, improve the efficiency of the simulations.

We argue that the great benefit of the integrated techniques is their potential of combining advantages and complementary features of the constituent basic VRTs. Thus, the integrated VRTs can achieve high efficiency gains even though the respective constituent basic VRTs do not operate under their optimal conditions, which is usually the case for a simulation scheme using a single VRT.

I. INTRODUCTION

Most often, estimation of a system’s key parameters is performed by a means of series of Monte Carlo (MC) experiments. The drawback of such a generic approach is that, in some cases, the required estimates are obtained very inefficiently. For instance, in the case of the bit-error-rate (BER) estimation of communication systems, the actual estimate is the outcome of the so-called “rare event” simulation problem, where an interesting outcome of an experiment occurs very rarely (e.g. simulations of multilevel and coded systems [1]). To illustrate the problem, consider the simulation of a (turbo–) coded system over an AWGN channel. The common performance measure, the BER, is usually in the order of $10^{-5}$ or lower. To obtain a relative error of 10% approximately $10^{4+2}$ bits need to be simulated [2]. Such low BER requirements are not unusual in today’s technology and sometimes even more stringent requirements are considered (e.g. by fiber optic communications, where BER in the order of $10^{-9}$ is a common figure [3]).

Fortunately, there exists a set of techniques, called variance reduction techniques (VRTs), which can significantly improve the efficiency of the standard MC method. Among them are: stratified sampling, importance sampling, correlated sampling, antithetic variables and control variates techniques [4]. Each can be successfully applied to a particular class of problems to reduce the variance of the standard MC estimate (e.g. [2]), and, in turn, to speed up the simulations. None is, however, general enough to be applicable to any arbitrary problem.

The complexity of implementing the techniques and the effort spent to understand them seems a challenging task in itself. This holds especially when, for the maximum performance, the techniques need to be applied under their (near-) optimal conditions. As a result, most applications of the VRTs are problem-specific, seriously limiting a wide spread of these techniques among researchers dealing with simulations.

We argue that by integrating several basic VRTs, more generic simulation schemes can be built allowing to simultaneously explore advantageous properties of the basic techniques involved. The integration allows to achieve high efficiency of the simulations, even though the constituent basic techniques operate in less stringent non-optimal conditions. Thus, instead of mastering a single VRT optimized for a certain system (problem), several basic techniques can be combined, conveniently utilizing the different properties of the basic techniques involved.

The general concept of constructing integrated VRTs (IVRTs), originally introduced by the authors in [5], is followed here. Suggested also in [6], the IVRTs can be observed in [1], [7] or in [8].

In this paper simulation results of a simple communication system are provided to validate and to verify the general concept of IVRTs.

II. THE SAMPLE MEAN ESTIMATION – THE COMMON SIMULATION PRACTICE

The problem of estimating error performance of a (digital) communication system can be mathematically presented as a problem of estimating the following probability [8]

$$P = P[X \in E] = \int_E p_X(x) \, dx,$$

(1)
where $P$ is the error performance measure, $X$ is the received information sequence, $E$ is some error region, and $p_X(\cdot)$ is the probability density function (PDF) of $X$. Using an indicator function defined as
\[
1_E(x) = \begin{cases} 
1 & \text{if} \ x \in E \\
0 & \text{if} \ x \notin E
\end{cases}
\] (2)
we can rewrite (1) to obtain
\[
P = \int_{-\infty}^{\infty} 1_E(x)p_X(x) \, dx.
\] (3)
A generalization of (3) leads to
\[
P = \int_A f(x)p_X(x) \, dx
\] (4)
which is essentially a formulation of the MC integration problem [4], where $A$ is the integration domain and $f(\cdot)$ denotes an arbitrary real-valued function, for instance an indicator function as defined in (2).

The form of (3) and (4) indicate that the problem of estimating the error performance of a communication system is, in fact, equivalent to the problem of performing the MC integration. Thus, MC methods are applicable. The most widely used of MC methods is the sample mean (SM) or the crude Monte Carlo method, which relies on a series of repetitive experiments of a system, each time providing a single estimate of $f(X)$.

The sample mean MC estimator is given by
\[
\hat{P} = \frac{1}{N} \sum_{i=1}^{N} f(X_i) \approx P, \quad X \sim p_X
\] (5)
where $N$ is the number of experiments (samples), `$\sim$’ denotes that the samples of a random variable $X$ are generated from the PDF $p_X$. The variance of the sample mean estimator is given by
\[
\text{var} [\hat{P}] = \frac{\text{var}[f(X)]}{N}.
\] (6)
Clearly, the variance of $\hat{P}$ is proportional to the variance of the random variable $f(X)$, and, in the standard MC method, is reduced by increasing the number of experiments. This is exactly the reason why in practice the experiments must be repeated a number of times to get “smooth” and reliable estimates.

In Fig. 1 a commonly used block diagram representing the standard MC procedure to simulate communication systems is presented.

III. BASIC VARIANCE REDUCTION TECHNIQUES
A. General Idea

Variance reduction techniques are the means intended to improve the efficiency of the crude MC method. They all are based on the common idea to exploit some degree of the knowledge of a problem (system) in order to reduce the variance of the estimator. They attempt to explore the design space existing between two extreme approaches: the first one of using analytical solution when having a perfect knowledge of the problem, and the second one of using crude Monte Carlo method while having no knowledge of the problem structure.

Formally written, they all are concerned with minimizing (6), and thus
\[
\min \left( \frac{\text{var}[f(X)]}{N} \right) \Leftrightarrow \begin{\{ \right)  \begin{align*}
\text{maximize} & \quad N \\
\text{minimize} & \quad \text{var}[f(X)].
\end{align*}
\] (7)

In contrast to the standard sample mean MC method, the VRTs are not concerned with increasing the number of experiments $N$ to reach a certain acceptable accuracy of an estimate. However, they minimize $\text{var}[f(X)]$, instead. Note that in the overall simulation procedure the VRTs are applied on top of the standard SM-MC method, so the scheme can take a real advantage of both options indicated in (7).

A general description of the considered basic VRTs can be found in [4], [9], [10] or in [5]. In the sequel of this section we present implementation aspects of the techniques in the context of their application to error performance evaluation of communication systems. Only three basic VRTs, relevant from the viewpoint of the considered integrated schemes, are presented.

B. Practical aspects

1) Stratified Sampling (SS): In the stratified sampling technique the whole region of interest is split into $M$ disjoint sub-regions (strata). Thus, the whole space of erroneous reception, $E_i$, is divided into a number of separate sub-regions, each representing a single error (whether a bit, a symbol, a word or a frame, referred throughout the paper as a “symbol”). Thus,
\[
E = \bigcup_{k=1}^{M} E_k, \quad \text{where} \quad E_j \cap E_l = \emptyset \quad \text{for} \quad l \neq j.
\] (8)
The number of experiments (samples) is then assigned to each stratum by some rule, according to the “importance” of that particular stratum. A common practice, used also here, is a proportional stratification, where the number of samples per $k^{th}$ stratum is calculated as
\[
N_k = P_k N,
\] (9)
and where $P_k$ is a relative probability of the $k^{th}$ stratum, as
\[ P_k = \int_{E_k} p_X(x) \, dx. \quad (10) \]

Essentially, the SS technique can be seen as a set of standard SM-MC procedures operating individually on each and every stratum with the respective number of experiments set to $N_k$.

From the practical point of view, the application of the proportional stratified sampling is relatively simple, which makes the approach and the technique especially attractive within the framework of IVRTs.

The necessary probabilities $P_k$ of each stratum are determined by providing an estimate of the stratum probability based on the Euclidian distance between the transmitted symbol and the error symbols constituting the strata as well as on a given channel model (see Sec. V).

The SS estimator is given by
\[ \hat{P}_{SS} = \sum_{k=1}^{M} P_k \frac{1}{N_k} \sum_{i=1}^{N_k} f(X_i) \approx P, \quad (11) \]
while its block diagram is presented in Fig. 2.

2) Importance Sampling (IS): The technique uses an alternative distribution function with the goal of concentrating the samples on “important” areas in the region of interest. The choice of the so-called “biased” density is thus crucial. In fact, the entire research in the field of importance sampling (IS) technique is a quest for near-optimal (and hence effective) biased distributions.

With reference to the framework of IVRTs, the requirement of the highest efficiency of a single technique is not relevant, as the performance of the overall simulation scheme is determined by the combination of several basic techniques. Therefore we use mean translation IS (MT-IS), where a biased density is obtained by transforming (shifting) a mean value of the original PDF [11].

The respective IS estimator of $P$ is provided by
\[ \hat{P}_{IS} = \frac{1}{N} \sum_{i=1}^{N} f(X_i) \frac{p_X(X_i)}{p_X^*(X_i)} \approx P, \quad X_i \sim p_X^* \quad (12) \]
where the samples are now taken from the biased distribution $p_X^*$. The block diagram representing the implementation of the IS estimator is presented in Fig. 3.

3) Antithetic Variables (AV): In its simplest form, the technique uses samples of two respective highly negatively correlated random variables at each single MC experiment run. The average of the two estimates is then provided as an estimate. In our context, it corresponds to the situation where the experiment with the first (original) sample $X_i$ does not yield an error so an antithetic sample, $\sim X_i$, is constructed (taking into account the system model), for which the error occurs.

The estimate based on the AV technique is unbiased, and the technique can provide desirable variance reduction when an effective antithetic variable can be constructed. This is usually the case, since an error boundary (placed usually half way the distance between symbols) can easily be used for this purpose.

The estimator based on the AV technique is given by
\[ \hat{P}_{AV} = \frac{1}{N} \sum_{i=1}^{N} f(X_i) + f(\sim X_i) \quad (13) \]
while its implementation is provided in Fig. 4.

IV. INTEGRATED VARIANCE REDUCTION TECHNIQUES

A. Stratified Sampling and Importance Sampling

This technique combines the SS and the IS techniques. First, the whole region of interest is divided into disjoint strata, each corresponding to erroneous symbol reception. The simulation is then performed by generating a random sample with a particular stratum in mind. After the stratum selection, the IS technique is deployed, which shifts the original PDF into the direction of the selected stratum. The MT-IS technique is used, so the original PDF is shifted half the distance between the transmitted symbol and the erroneous symbol constituting the selected stratum.

Schematically, the simulation scheme, referred here to as “SS + IS” scheme is presented in Fig. 5. We note that this is, in fact, the technique used in [1], [7] and in [8].

B. Stratified Sampling, Antithetic Variables and Importance Sampling

The scheme, denoted here as “SS + AV + IS” augments the previously discussed example of the IVRT by incorporating the AV technique. The augmented simulation scheme works as follows. First, exactly as before, the stratification is performed. Then a random sample is generated, for which an antithetic sample is calculated taking into account the selected stratum.
Finally, both samples (the original and the antithetic) are biased with the application of the IS technique, the system function is performed and the average of the two outcomes is provided. The scheme is depicted in Fig. 6.

V. EXAMPLE: M-PSK OVER AWGN

To illustrate and validate the concept, we evaluate symbol-error-rate (SER) estimation for M-PSK modulation in an AWGN channel. Although an analytical solution for the error performance evaluation of such a system is available, the choice of the setup is motivated by its simplicity as well as its versatility. It gives a clear but yet instructive insight into various aspects of practical applications of IVRTs. We argue that the concept and the presented examples of the IVRTs can easily be extended to more practical cases of much complex systems.

The simulation procedure complies with the general scheme of a communication chain as presented in Fig. 1 (lower part), where the source is assumed to produce all–zero bit sequence, the transmitter is the 8-PSK modulator, the channel is the AWGN channel and the receiver is the conventional 8-PSK demodulator, respectively. The estimate represents the SER estimation, which indicates the erroneous reception of the transmitted symbol.

A. Simulation Schemes Used

- **Standard MC** simulation scheme following directly the diagram presented in Fig. 1;
- **IS** scheme, where a single basic VRT is used,
- **SS+IS** integrated scheme, which follows the description given in Sec. IV-A and Fig. 5. The corresponding strata probabilities, $P_k$, are calculated according to [1]

$$ P_k = \frac{Q\left(\frac{d_k}{\sigma}\right)}{\sum_{k=1}^{M} Q\left(\frac{d_k}{\sigma}\right)} $$

where $d_k$ is the euclidian distance between the transmitted symbol and the erroneous symbol constituting a stratum, $Q(\cdot)$ is the $Q$-function, and $\sigma$ is the standard deviation of the channel Gaussian process;
- **SS+AV+IS** integrated scheme, which follows the description given in Sec. IV-B and Fig. 6. The corresponding strata probabilities are calculated using (14).

The simulations using the considered schemes are performed with two goals: the first one to show the variance reduction capabilities of a scheme, where the number of experiments, $N$, is fixed, and the second one to provide the actual simulation time needed to obtain the estimates, where the number of experiments $N$ is not fixed a priori, and the simulations are concluded when a certain predefined accuracy (maximum acceptable relative error) is reached.

B. Simulation Results

We start with the SER plotted against SNR presented Fig. 7, which demonstrates that all the considered simulation schemes provide the same unbiased SER estimates. In fact, for each scheme, the simulations are halted whenever the relative error is not higher than 10%. An excellent match between the results of all the schemes is clear.

To compare the variance reduction capabilities, in Fig. 8 the relative error (normalized standard deviation) of the estimate of each scheme is plotted as a function of SER. The results are obtained for a fixed number of experiments. The following conclusions can be drawn. For medium, low and very low SERs, all the schemes incorporating VRTs perform significantly better than a standard MC method leading to lower estimation errors. Only at very high (and in this case impractical) SERs, the standard MC method gives better results. The variance reduction capabilities of the schemes using IVRTs
are higher than of the scheme using only a single VRT with the respective settings. Furthermore, the scheme “SS+AV+IS” incorporating 3 basic VRTs outperforms the “SS+IS” scheme using only 2 basic VRTs, suggesting that sensible integration of an arbitrary number of basic VRTs can be beneficial.

Next, we concentrate on the actual CPU performance of the schemes. In Fig. 9, a plot showing the CPU utilization as a function of the SER is presented. The interpretation closely follows the discussion of Fig. 8. Again, all the schemes incorporating the VRTs outperform the standard MC approach for moderate, low and very low error probabilities. The performance of the schemes incorporating VRTs matches the pattern indicated before, where “SS+AV+IS” outperforms “SS+IS”, which, in turn, outperforms the basic scheme of “IS”. This holds even though both presented schemes of IVRTs suffer from increased complexity (more techniques combined) and, thus, increased CPU load per single experiment. The results indicate that IVRTs can, indeed, be used to improve the efficiency of the MC simulations.

VI. CONCLUSIONS

In this paper the performance of the IVRTs applied in the context of error performance estimation of communication systems is studied. The general framework of IVRTs introduced in [5] is followed, where the integrated techniques are constructed using a number of basic VRTs. Thus, instead of optimizing a single technique, an approach is taken where several techniques operating under less strict requirements are combined.

Using two examples of the IVRTs, where SS is combined with IS, and with IS and AV, respectively, the concept is demonstrated and evaluated. The SER estimation results for M-PSK modulation in the AWGN channel are provided for verification. The results clearly demonstrate that the variance reduction gain of a simulation scheme can be increased when several basic VRTs are combined. Consequently, it is concluded that IVRTs can, indeed, lead to increased efficiency of simulations.

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