

Bounds and algorithm for direction finding of phase modulated signals

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Abstract

In many cases where direction finding is of interest, the signals impinging on an antenna array are known to be phase modulated, hence to have a constant modulus (CM). This is a strong property, by itself already sufficient for source separation, and can be used to construct improved direction finding algorithms. We first present the relevant Cramer-Rao bounds for arbitrary array configurations, and arbitrary number of signals and then specialize to uniform linear arrays with a single source. We then propose a simple suboptimal direction estimation algorithm, in which the signals are separated using the CM property only. Compared to the ESPRIT algorithm and to the Cramer-Rao bound for arbitrary signals, the algorithm shows good results.

1. Introduction

Direction-of-arrival (DOA) estimation of multiple signals impinging on an antenna array is a well-studied problem in signal processing. “Traditional” methods exploit knowledge of the array manifold or its structure without using information on the signals. Examples for such algorithms are MUSIC [5], ESPRIT [4], MLE [12], and WSF [10]. For signals with known waveforms an algorithm is derived in [1]. Other, ‘blind’ methods exploit properties of the signals such as non-gaussianity [3] or cyclostationarity [11]. These methods are more robust to array manifold errors due to the extra information they use. Although phase modulated signals are ubiquitous in the communication field, no detailed study of the exploitation of the constant modulus property for multiple source DOA estimation has been done so far. As we show here, a large improvement can be achieved by exploiting this information.

Since the pioneering work of Treichler and Agee [8], it is known that the constant modulus (CM) property is a strong property, by itself already sufficient for source separation. After separation of the signals, the DOA estimation problem is decoupled and can be

done for each source individually. Such a scheme is proposed in [6], where the CM signals are sequentially separated using the so-called CM array. Weak points of this and related iterative CM algorithms are their initialization, the problematic recovery of all signals, and their unpredictable convergence, which may require several hundred samples per signal. To counter these problems, Mathur et al. [2] propose to initialize each stage of the algorithm by a weight vector found by the MUSIC algorithm. However, it is well known that sequential DOA estimation yields poor performance for the weak sources when the stronger sources are not completely removed.

Recently, Van der Veen and Paulraj [9] have found an analytic solution to the CM source separation problem, in which all weight vectors are found simultaneously and reliably, from a small number of samples and without initialization problems. Thus, this algorithm is very attractive to be used as a first step in the DOA estimation problem. Moreover, it is applicable to any array geometry. Knowing that there is a good algorithm, it becomes interesting to study the performance bounds for DOA estimation of CM signals. Here, our aim is to present such bounds. We also give explicit bounds for the signal phase estimates.

We demonstrate by simulations that the proposed algorithm almost achieves the CRB for constant modulus signals, which is below the bound for arbitrary signals. Hence the algorithm outperforms any algorithm which does not use the CM property. We also demonstrate the robustness of the algorithm to various model errors.

2. Data model

Consider an array with p sensors receiving q narrow-band constant modulus signals. Under standard assumptions, we can describe the received signal as an instantaneous linear combination of the source signals, i.e.,

$$\mathbf{x}(t) = \mathbf{A}\mathbf{B}\mathbf{s}(t) + \mathbf{n}(t) \quad (1)$$

where

$\mathbf{x}(t) = [x_1(t), \dots, x_p(t)]^T$ is a $p \times 1$ vector of received signals at time t ,

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$\mathbf{A} = \mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_q)]$, where $\mathbf{a}(\theta)$ is the array response vector for a signal from direction θ , and $\boldsymbol{\theta} = [\theta_1, \dots, \theta_q]$ is the DOA vector of the sources,

$\mathbf{B} = \text{diag}(\boldsymbol{\beta})$ is the channel gain matrix, with parameters $\boldsymbol{\beta} = [\beta_1, \dots, \beta_q]^T$, where $\beta_i \in \mathbb{R}^+$ is the amplitude of the i -th signal as received by the array,

$\mathbf{s}(t) = [s_1(t), \dots, s_q(t)]^T$ is a $q \times 1$ vector of source signals at time t ,

$\mathbf{n}(t)$ is the $p \times 1$ additive noise vector, which is assumed spatially and temporally white gaussian distributed with covariance matrix $\nu \mathbf{I}$, where $\nu = \sigma^2$ is the noise variance.

In our problem, the array is assumed to be calibrated so that the array response vector $\mathbf{a}(\theta)$ is a known function. As usual, we require that the array manifold satisfies the uniqueness condition, i.e., every collection of p vectors on the manifold are linearly independent.

We further assume that all sources have constant modulus. This is represented by the assumption that for all t , $|s_i(t)| = 1$ ($i = 1, \dots, q$). Unequal source powers are absorbed in the gain matrix \mathbf{B} . Phase offsets of the sources after demodulation are part of the s_i . Thus we can write $s_i(t) = e^{j\phi_i(t)}$, where $\phi_i(t)$ is the unknown phase modulation for source i , and we define $\boldsymbol{\phi}(t) = [\phi_1(t), \dots, \phi_q(t)]^T$ as the phase vector for all sources at time t .

Finally, we assume that N samples $[\mathbf{x}(1), \dots, \mathbf{x}(N)]$ are available.

3. Cramer-Rao bounds

The Cramer-Rao bound (SRB) provides a lower bound on parameter estimation variance for any unbiased estimator. We present Cramer-Rao bounds for DOA and signal phase estimation of multiple CM signals, postponing the derivations to the full version of the paper.

The likelihood function is given by

$$L(\mathbf{x}|\mathbf{s}, \boldsymbol{\theta}, \boldsymbol{\beta}, \nu) = c \exp \left\{ -\frac{1}{\nu} \sum_{k=1}^N \|\mathbf{x}(k) - \mathbf{A}\mathbf{B}\mathbf{s}(k)\|^2 \right\}$$

where $c = \frac{1}{(2\pi)^{N(\frac{p}{2})^{pN}}}$.

Let $\mathcal{L}(\mathbf{x}|\mathbf{s}, \boldsymbol{\theta}, \boldsymbol{\beta}, \nu) = \log L(\mathbf{x}|\mathbf{s}, \boldsymbol{\theta}, \boldsymbol{\beta}, \nu)$. After omitting constants we obtain

$$\mathcal{L}(\mathbf{x}|\mathbf{s}, \boldsymbol{\theta}, \boldsymbol{\beta}, \nu) = -pN \log \nu - \frac{1}{\nu} \sum_{k=1}^N \|\mathbf{x}(k) - \mathbf{A}\mathbf{B}\mathbf{s}(k)\|^2.$$

Following [7], the estimation of the noise variance is decoupled from all other parameters, and its bound

can be computed separately as

$$\text{CRB}_N(\nu) = \frac{\nu^2}{pN}.$$

The remaining parameters are collected in the vector $[\boldsymbol{\phi}(1)^T, \dots, \boldsymbol{\phi}(N)^T, \boldsymbol{\theta}^T, \boldsymbol{\beta}^T]^T$. Define

$$\mathbf{S}_k = \text{diag}(\mathbf{s}(k)) \quad \text{and} \quad \mathbf{D} = \left[\frac{d\mathbf{a}}{d\boldsymbol{\theta}}(\theta_1), \dots, \frac{d\mathbf{a}}{d\boldsymbol{\theta}}(\theta_q) \right].$$

The Fisher information matrix associated to the estimation of the parameter vector can be derived as

$$\mathbf{FIM}_N = \left[\begin{array}{cc|cc} \mathbf{H}_1 & 0 & \boldsymbol{\Delta}_1^T & \mathbf{E}_1^T \\ & \ddots & \vdots & \vdots \\ 0 & \mathbf{H}_N & \boldsymbol{\Delta}_N^T & \mathbf{E}_N^T \\ \hline \boldsymbol{\Delta}_1 & \cdots & \boldsymbol{\Delta}_N & \boldsymbol{\Gamma} \\ \mathbf{E}_1 & \cdots & \mathbf{E}_N & \boldsymbol{\Lambda} \end{array} \right] \quad (2)$$

where

$$\begin{aligned} \mathbf{H}_k &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} \right)^T = \frac{2}{\nu} \text{Re}(\mathbf{S}_k^* \mathbf{B}^* \mathbf{A}^* \mathbf{A} \mathbf{B} \mathbf{S}_k) \\ \boldsymbol{\Delta}_k &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} \right)^T = -\frac{2}{\nu} \text{Im}(\mathbf{S}_k^* \mathbf{B}^* \mathbf{D}^* \mathbf{A} \mathbf{B} \mathbf{S}_k) \\ \mathbf{E}_k &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} \right)^T = -\frac{2}{\nu} \text{Im}(\mathbf{S}_k^* \mathbf{A}^* \mathbf{A} \mathbf{B} \mathbf{S}_k) \\ \boldsymbol{\Gamma} &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \right)^T = \frac{2}{\nu} \sum_{k=1}^N \text{Re}(\mathbf{S}_k^* \mathbf{B}^* \mathbf{D}^* \mathbf{D} \mathbf{B} \mathbf{S}_k) \\ \boldsymbol{\Lambda} &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} \right)^T = \frac{2}{\nu} \sum_{k=1}^N \text{Re}(\mathbf{S}_k^* \mathbf{A}^* \mathbf{A} \mathbf{B} \mathbf{S}_k) \\ \boldsymbol{\Upsilon} &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \right)^T = \frac{2}{\nu} \sum_{k=1}^N \text{Re}(\mathbf{S}_k^* \mathbf{A}^* \mathbf{A} \mathbf{S}_k) \end{aligned}$$

\mathbf{H}_k^{-1} would be the CRB on the estimation of the unknown source phases at time k , in case the DOAs and amplitudes are known. Similarly, $\boldsymbol{\Gamma}^{-1}$ and $\boldsymbol{\Upsilon}^{-1}$ provide bounds on the estimation of the DOAs and amplitudes, respectively, when other parameters are known. The matrices $\boldsymbol{\Delta}_k$, \mathbf{E}_k and $\boldsymbol{\Lambda}$ represent the couplings between the parameters.

The bounds on the individual parameters are obtained after inversion of the Fisher information matrix. This can be carried out in block-partitioned form (using Schur complement formulas and the Woodbury identity) which leads to more explicit expressions. Thus, assuming that the \mathbf{H}_k are invertible, let

$$\begin{aligned} \boldsymbol{\Xi}_{11} &= \sum_{k=1}^N \boldsymbol{\Delta}_k \mathbf{H}_k^{-1} \boldsymbol{\Delta}_k^T \\ \boldsymbol{\Xi}_{21} &= \sum_{k=1}^N \mathbf{E}_k \mathbf{H}_k^{-1} \boldsymbol{\Delta}_k^T \\ \boldsymbol{\Xi}_{12} &= \sum_{k=1}^N \boldsymbol{\Delta}_k \mathbf{H}_k^{-1} \mathbf{E}_k^T \\ \boldsymbol{\Xi}_{22} &= \sum_{k=1}^N \mathbf{E}_k \mathbf{H}_k^{-1} \mathbf{E}_k^T \end{aligned}$$

and define the $2q \times 2q$ matrix

$$\boldsymbol{\Psi} = \left[\begin{array}{cc} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} \end{array} \right] = \left[\begin{array}{cc} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^T \\ \boldsymbol{\Lambda} & \boldsymbol{\Upsilon} \end{array} \right] - \left[\begin{array}{cc} \boldsymbol{\Xi}_{11} & \boldsymbol{\Xi}_{12} \\ \boldsymbol{\Xi}_{21} & \boldsymbol{\Xi}_{22} \end{array} \right].$$

Using the Schur complement formula twice, the CRB for DOAs and amplitudes can be written more explicitly as

$$\begin{aligned} \text{CRB}_N(\boldsymbol{\theta}) &= \text{diag} \left[\boldsymbol{\Psi}_{11} - \boldsymbol{\Psi}_{12} \boldsymbol{\Psi}_{22}^{-1} \boldsymbol{\Psi}_{21} \right]^{-1} \\ \text{CRB}_N(\boldsymbol{\beta}) &= \text{diag} \left[\boldsymbol{\Psi}_{22} - \boldsymbol{\Psi}_{21} \boldsymbol{\Psi}_{11}^{-1} \boldsymbol{\Psi}_{12} \right]^{-1}. \end{aligned} \quad (3)$$

Similarly, using the Woodbury identity, the bound on the estimation variance of the signal phases follows as

$$\text{CRB}_N(\phi(k)) = \text{diag} \left\{ \mathbf{H}_k^{-1} \left[\mathbf{I} + [\Delta_k^T \quad \mathbf{E}_k^T] \Psi^{-1} \begin{bmatrix} \Delta_k \\ \mathbf{E}_k \end{bmatrix} \mathbf{H}_k^{-1} \right] \right\}. \quad (4)$$

Note that the number of samples and the quality of DOA estimation affect the phase bound only through the matrix Ψ^{-1} .

Single CM source

To obtain more insight into the CRBs, we consider the case of DOA estimation of a single CM source. The CRB on the DOA in this case is given by

$$\text{CRB}_N(\theta) = \frac{1}{2N \text{SNR} \|\mathbf{P}_{\mathbf{a}(\theta)}^\perp \mathbf{d}(\theta)\|^2}$$

where $\mathbf{d}(\theta) = \frac{d\mathbf{a}(\theta)}{d\theta}$, $\mathbf{P}_{\mathbf{a}}^\perp = \mathbf{I} - \mathbf{a}(\mathbf{a}^*\mathbf{a})^{-1}\mathbf{a}^*$, and $\text{SNR} = \beta^2/\nu$. We can also prove that the phase estimation variance is given by

$$\text{CRB}_N(\phi(k)) = \frac{1}{2 \text{SNR} \|\mathbf{a}(\theta)\|^2} \left(1 + \frac{1}{N} c(\theta) \right)$$

where

$$c(\theta) = \frac{(\text{Im}(\mathbf{d}(\theta)^* \mathbf{a}(\theta)))^2}{\|\mathbf{a}(\theta)\|^2 \|\mathbf{P}_{\mathbf{a}(\theta)}^\perp \mathbf{d}(\theta)\|^2}.$$

$c(\theta)$ represents the effect of channel estimation error on the signal phase estimation.

Further simplification of the above bounds is possible if we assume that the antenna array is a uniform linear array (ULA) with antennas spaced by d wavelengths. In this case,

$$\begin{aligned} \text{CRB}_N(\theta) &= \frac{6}{p(p^2 - 1)N \text{SNR} (2\pi d)^2 \cos^2(\theta)} \\ \text{CRB}_N(\phi(k)) &= \frac{1}{2p \text{SNR}} \left[1 + \frac{3}{N} \frac{p-1}{p+1} \right]. \end{aligned}$$

Note that the estimation quality of the signal phases is independent of the antenna spacing and the DOA, and quickly becomes independent of the number of samples N .

4. CM-DOA estimation algorithm

A suboptimal but simple algorithm to estimate the DOAs using the CM property is to

1. Blindly estimate a matrix $\hat{\mathbf{A}} = [\hat{\mathbf{a}}_1, \dots, \hat{\mathbf{a}}_q]$ using the CM assumption,
2. For each column $\hat{\mathbf{a}}_i$ of $\hat{\mathbf{A}}$, estimate the direction $\hat{\theta}_i$ which fits best.

A closed-form solution for the first step is provided by the ACMA algorithm. It is described in detail in [9] and will not be discussed here. The second step is known to be a one dimensional projection of each $\hat{\mathbf{a}}_i$ onto the array manifold, given by

$$\hat{\theta}_i = \arg \max_{\theta} \frac{|\hat{\mathbf{a}}_i^* \mathbf{a}(\theta)|}{\|\mathbf{a}(\theta)\|} \quad (5)$$

The computational complexity of this method is comparable to other DOA estimation methods. The first step involves $9q^4N + 36p^2N$ real flops when the Schur subspace decomposition version is used [9]. The second step just consists of a sequence of q one-dimensional searches over the parameter space, and its complexity is marginal relative to the first step for larger values of q . Note that for large arrays where $q^2 \leq p$ the dominant term is $36p^2N$, which is the same order of magnitude as the computation of a data covariance matrix in any eigenspace-based method. Also note that compared to multidimensional search methods, the CM-DOA is very attractive.

The advantage of this CM-DOA algorithm is that it is applicable to arbitrary array configurations, unlike other fast methods such as the ESPRIT which exploits a specific array structure and breaks down with multipath propagation. Although suboptimal, its estimates are usually quite close to the CRB.

5. Simulation results

It is interesting to compare the DOA Cramer-Rao bounds for CM signals versus the usual case of arbitrary signals [7], and versus the case of known signals with unknown amplitudes (see [1]). Because of the complex nature of the expressions, this is practical only graphically for specific examples. We also compare the CM-DOA algorithm to ESPRIT by means of simulations.

We have used a $p = 8$ element ULA with spacing $d = \frac{1}{2}$ wavelength and $q = 2$ equipowered random phase CM signals. If not specified otherwise, we took $N = 50$ samples, $\text{SNR} = 20$ dB, first source located at 0° (boresight), second at 5° . The results have been averaged over 400 Monte-Carlo runs.

We have carried out three simulation cases:

1. *Varying source separation, from 2° to 20° .* As seen in figure 1(a), the CM-DOA algorithm is very close to its CRB and hence outperforms any DOA estimation which does not use the CM information.
2. *Varying SNR.* See figure 1(b). We see that the CM-DOA estimator almost achieves the CRB.
3. *Varying array model mismatch.* We have corrupted the entries of the array response vectors

by white gaussian noise with variance -50 dB to 0 dB relative to the array manifold. The DOA estimation variance for the first source is presented in figure 1(c).

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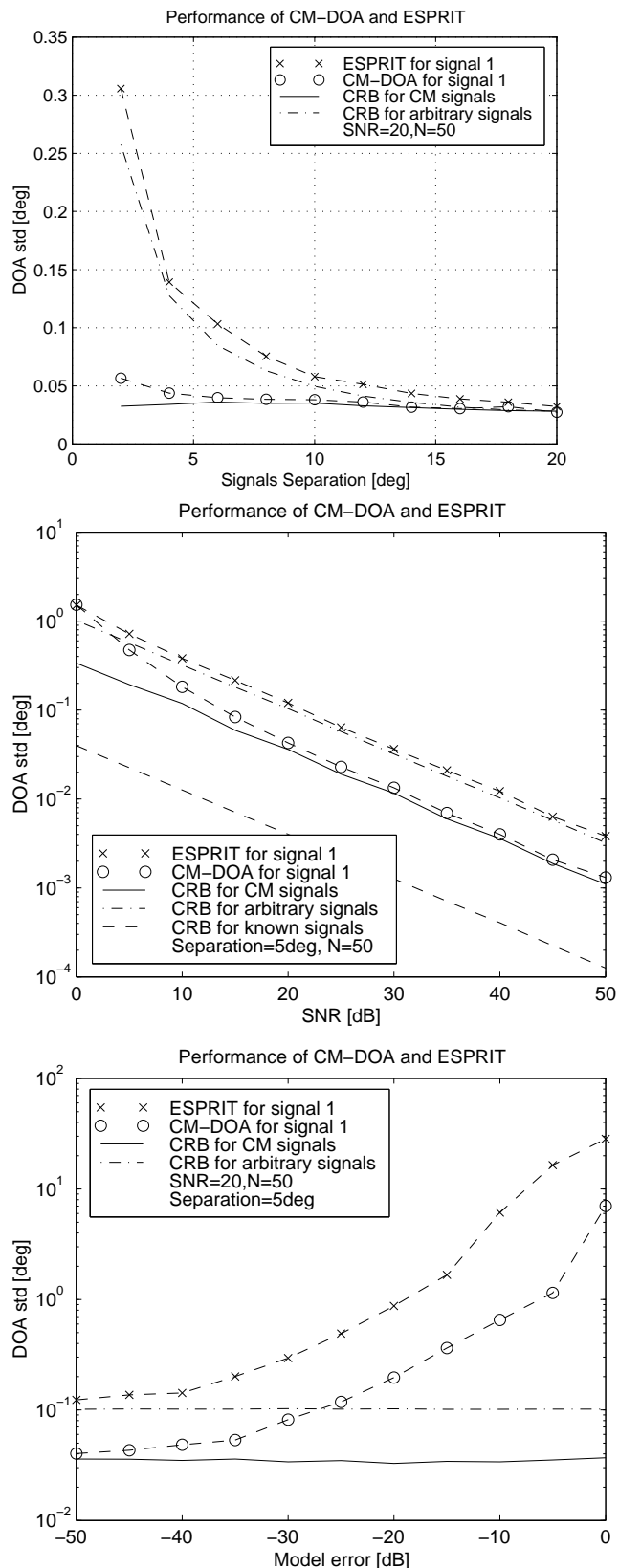


Figure 1. DOA estimation accuracy for CM-DOA and ESPRIT (a) varying source separation, (b) varying SNR. (c) model errors.