On Spectral Factorization and Riccati Equations for Time-Varying Systems in Discrete Time

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Abstract

It is known that positive operators Ω on a Hilbert space admit a factorization of the form $\Omega = W^*W$, where W is an outer operator whose matrix representation is upper. As upper Hilbert space operators have an interpretation of transfer operators of linear time-varying systems in discrete time, this proves the existence of a spectral factorization for time-varying systems. In this paper, the above result is translated from operator theory into control theory language, by deriving how such a factorization can be actually computed if a state realization of the upper part of Ω is known. The crucial step in this algorithm is the solution of a Riccati recursion with time-varying coefficients. It is shown that, under conditions, positive solutions exists, and that the smallest positive solution lead to a factor which is outer ('minimum-phase'). The outer factor can be computed numerically in a number of cases, e.g., if the system is initially time-invariant, periodically time-varying. More generally, for strictly stable systems it is shown that the Riccati recursion, when started from zero initial conditions, will strongly converge to the exact smallest positive solution, so that the outer factor can be computed in arbitrary precision for any finite interval in time. The results can also be formulated in terms of a time-varying positive real lemma. Finally, some connections are provided with Riccati equations that occur in related problems in time-varying systems theory, such as inner-outer factorization, orthogonal embedding and the time-varying bounded-real lemma.

Keywords

Spectral factorization, time-varying system theory, positive real lemma, Riccati equations (discrete-time, time-varying; convergence), outer factors.

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1. INTRODUCTION

In modern system and (H_∞)-control theory, for example linear quadratic optimal control, optimal filtering and sensitivity minimization, Riccati equations play an important role. Such equations are, for discrete-time systems, of the form

$$P = A^*PA + Q - \left[C + B^*PA\right]^* (R + B^*PB)^{-1} \left[C + B^*PA\right] \quad (1)$$

where P is unknown. There is a family of related forms of this type of equation, obtained by making certain substitutions, and the precise form depends on the application. Underlying these problems is typically a spectral factorization problem, and the discrete-time Riccati equation corresponding to this problem has originally been studied in [3, 4]. The equation usually has more than one solution P, and important issues are the computation of the largest or smallest Hermitian solutions, as these correspond to minimal-phase properties of spectral factors, or to the stability of (closed loop) transfer operators constructed from the solution. Such solutions are, for time-invariant systems, obtained by an analysis of the eigenvalues and invariant subspaces of an associated (Hamiltonian) matrix. A recent overview of solution methods, as well as many references to older literature, can be found in the collection [9].

For general time-varying systems, the equation (1) becomes a recursion

$$P_{k+1} = A_k^*P_kA_k + Q_k - \left[C_k + B_k^*P_kA_k\right]^* (R_k + B_k^*P_kB_k)^{-1} \left[C_k + B_k^*P_kA_k\right] \quad (2)$$

in which $\{A_k, B_k, C_k, D_k\}_{-\infty}^{\infty}$ is a time-varying realization, and where the dimensions of the matrices can also be time-varying: the number of inputs, outputs and states need not be constant in time. For this case, much less is known on the structure of solutions. One reason is that the usual eigenvalue analysis to classify stable and unstable systems is no longer applicable, e.g. because instead of a single A, we now have to consider the spectral properties of products of the A_k, and A_k need not be square. Attempts have been made to define the notions of 'time-varying' poles and zeros (e.g., [2, 15]) but these results are not general enough to cope with a time-varying number of poles.

In this paper, we approach the time-varying Riccati equation from a different angle, by starting from certain standard factorization problems (such as spectral factorization and inner-outer factorization) of operators in a Hilbert space. The same approach is followed in [16], although in that paper, the starting point is the existence of the Cholesky factor of a positive definite, finite size matrix. The Riccati recursion in these factorization problems results once a state realization for the operator is assumed. Solutions of the spectral factorization and inner-outer factorization problems are known also in the more general case of Hilbert space nest algebras (see the work of Arveson [8]), and this context applies to time-varying systems, too. For example, a bounded positive operator Ω has a factorization into

$$\Omega = W^*W$$

where W can be chosen to be outer ('minimum-phase'). We will show how, from this property of W, properties on the related time-varying Riccati equation can be derived. In particular, the fact that there exists an outer factor W will imply the existence of a smallest positive solution to the Riccati equation.

An alternative route is to study spectral factorization problems via optimal control techniques. For the continuous-time case, this was done by Anderson et al. [6].

One of the purposes of this paper is to show how a recently developed compact notation for time-varying systems (see [12, 13]) provides for a very straightforward and elegant derivation of important properties of spectral factors and the corresponding Riccati equations in terms of state space quantities. The direct analysis of Riccati equations with time-varying parameters is believed to be much harder and quite tedious.

One application in which (time-varying) spectral factorizations play a role is in the solution of the problem of the uniformly optimal control of time-varying systems, as discussed by Feintuch and Francis [14]. In this problem, a plant is given, and a causal regulator is to be designed such that the closed-loop system is stable and certain noise terms, acting on the inputs of the plant and the regulator, have minimal effect on the outputs of the closed-loop system. Their solution to this problem involves, besides inner-outer factorizations and the solution to a Nehari problem, also two spectral factorizations [14]. This solution was described at the operator level, and it was remarked that "at present, computation of uniformly optimal controllers for time-varying systems is not feasible". Currently however, with algorithms for the inner-outer factorization [17], the Nehari problem [13] and spectral factorization (this paper) available at a state-space level, computation of the optimal controller is in principle possible, once a state realization of the plant is known. The resulting algorithm consists of a number of Riccati recursions that run both forward and backward in time. The occurrence of a backward recursion implies that optimal controllers can only be computed if the realization of the plant is known for all time (for else approximating windowing schemes must be used). In this line, one possible application that is feasible at present is the computation of an optimal controller for a plant whose parameters are shifted from one operating point to another.

Other, related factorization problems in which a time-varying Riccati equation arises are the inner-outer factorization problem of bounded 'causal' operators T,

$$T = UT_o\,,$$

where U is an isometry ($U^*U = I$) and T_o is outer, and the orthogonal embedding problem of contractive causal operators T (also known as the unitary extension problem),

$$\text{Given } T \text{, find an inner system } \Sigma \text{ such that } \Sigma = \begin{bmatrix} T & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

There is a connection between the solution of the embedding problem and the time-variant bounded real lemma [7], which is well known in its time-invariant version. There is also a connection between the spectral factorization problem and the time-variant positive real lemma (see e.g., [11] for the time-invariant discrete-time version). Our approach to the embedding problem is published in [21] and submitted for publication in [20], while results on the inner-outer factorization appeared in [17].

In this paper, we will be mainly concerned with the spectral factorization problem. We will only consider the 'easy' case where the inverted term in the Riccati equation exists and is bounded, and in particular where this term is definite. Generalizations are still possible but analytically more difficult as it leads to generalized inverses with range conditions. The spectral factorization problem is treated in section 3, and the (related) time-varying version of the positive real lemma is formulated. Some computational issues are discussed in section 4. It is argued that (under conditions) the Riccati recursion converges to the exact smallest positive solution even if the recursion is started from an approximate initial point, $P_0 = 0$. This allows to compute spectral factors of general time-varying positive operators, even if they are not constant or periodical before some point in time. (The proof of this is relayed to an appendix.) Finally, in section 5, we discuss some connections with related problems in which a Riccati equation occurs, in particular the orthogonal embedding problem of contractive operators and the inner-outer factorization problem.

2. NOTATION AND PRELIMINARIES

Spaces

We consider time-varying transfer operators as bounded Hilbert space operators on ℓ_2-sequences. Such operators have an (infinite) matrix description

$$T = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \cdots \\ & T_{0,-1} & \boxed{T_{0,0}} & T_{0,1} & \\ \cdots & T_{1,-1} & T_{1,0} & T_{1,1} & \cdots \\ & \vdots & & \vdots & \end{bmatrix}$$

(the square identifies the $(0,0)$-th entry), where we will allow, for generality, that the entries $T_{i,j}$ are matrices themselves, say $T_{i,j} \in \mathbb{C}^{M_i \times N_j}$. To describe such operators properly, let $M = [\cdots, \boxed{M_0}, M_1, \cdots]$ be a sequence of non-negative integers. The space of non-uniform sequences ('signals') u such that the i-th entry of the sequence is an M_i-dimensional vector is denoted by

$$\mathcal{M} \;=\; \cdots \times \mathcal{M}_0 \times \mathcal{M}_1 \times \cdots\,,$$

where $\mathcal{M}_i = \mathbb{C}^{M_i}$. In this context, we call M an index sequence and write $\mathcal{M} = \mathbb{C}^M$ and $M = \#\mathcal{M}$. The space $\ell_2^{\mathcal{M}}$ is the space of sequences in $\mathcal{M}$ of finite 2-norm (bounded energy). This space is a Hilbert space. We denote by $\mathcal{X}(\mathcal{M},\mathcal{N})$ the space of bounded operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$. To be consistent with earlier literature in which this notation was defined [12, 19, 20, 13], we think of sequences as row vectors, and of operators as acting on the sequences at the left, so that we will write uT rather than Tu. An operator is said to be upper if its matrix representation is an upper matrix: $T_{i,j} = 0$ $(i > j)$, and we denote the space of bounded upper operators by $\mathcal{U}(\mathcal{M},\mathcal{N})$. Likewise, the space $\mathcal{L}$ of bounded lower operators, and the space $\mathcal{D} = \mathcal{L} \cap \mathcal{U}$ of bounded diagonal operators is defined. We will allow that some (or all but a finite) number of entries of index sequences are equal to zero. In this way, finite matrices are incorporated in the same framework.

An operator $T \in \mathcal{X}$ can be viewed as a time-varying transfer operator: its i-th row contains the impulse response of the system for an impulse at time i. An operator $T \in \mathcal{U}$ is said to be *causal*, because when $y = uT$ is the response of an input u which is zero up till time instant k, then y is also zero up till this point. If T is a time-invariant system, then its matrix representation has a Toeplitz structure.

The causal bilateral shift-operator on sequences is denoted by Z: it is such that

$$[\cdots \quad \boxed{u_0} \quad u_1 \quad \cdots]Z = [\cdots \quad \boxed{u_{-1}} \quad u_0 \quad \cdots].$$

If $u \in \mathcal{M}$, then we write $uZ \in \mathcal{M}^{(1)}$, where $\mathcal{M}^{(1)}$ is equal to the space sequence $\mathcal{M}$, shifted over one position. We will also need a diagonal shift operator: the k-th diagonal shift of $A \in \mathcal{X}$ is $A^{(k)} = Z^{*k}AZ^k$, and will shift the entries of A over k positions into the South-East direction: $(A^{(k)})_{i,j} = A_{i-k,j-k}$.

We will say that a Hermitian operator $X \in \mathcal{X}(\mathcal{M},\mathcal{M})$ is uniformly positive, $X \gg 0$, if

$$\exists\, \varepsilon > 0 \ : \quad \| uXu^* \| > \varepsilon \| uu^* \| \,, \text{for all } u \in \ell_2^{\mathcal{M}} .$$

If X is uniformly positive, then it is boundedly invertible.

In time-varying systems theory, one often applies a collection of input signals to a system. In view of this, it is useful to stack a collection of ℓ_2-sequences onto each other into a single larger operator, whose rows then correspond to the individual ℓ_2 sequences. Transfer operators T can be applied to such a collection of signals if the total energy of the collection is bounded. To translate this concept into mathematical terms, we define the Hilbert-Schmidt norm of an operator A in $\mathcal{X}$ as

$$\| A \|_{HS}^2 = \sum_{i,j=-\infty}^{\infty} \|A_{i,j}\|_F^2$$

and the Hilbert-Schmidt space of operators as

$$\mathcal{X}_2(\mathcal{M},\mathcal{N}) = \{X \in \mathcal{X}(\mathcal{M},\mathcal{N}) : \| X \|_{HS} < \infty\} .$$

Standard subspaces of $\mathcal{X}_2$ are $\mathcal{U}_2 = \mathcal{X}_2 \cap \mathcal{U}$, $\mathcal{L}_2 = \mathcal{X}_2 \cap \mathcal{L}$, $\mathcal{D}_2 = \mathcal{X}_2 \cap \mathcal{D}$, consisting of those elements of $\mathcal{X}, \mathcal{U}, \mathcal{L}, \mathcal{D}$ for which the norms of the entries are square summable. A stacked collection of signal sequences in $\ell_2^{\mathcal{M}}$ with total bounded energy then is an element of $\mathcal{X}_2(\mathbb{C}^{\mathbf{Z}}, \mathcal{M})$ [12]. We will use the shorthand $\mathcal{X}_2^{\mathcal{M}}$ for $\mathcal{X}_2(\mathbb{C}^{\mathbf{Z}}, \mathcal{M})$, although we will sometimes write $\mathcal{X}_2$ if the precise form of $\mathcal{M}$ is not of interest. Throughout this paper, elements of $\mathcal{X}_2^{\mathcal{M}}$ will have the interpretation as generalized input sequences, rather than as operators in $\mathcal{X}$. We will often use the space decomposition

$$\mathcal{X}_2 = \mathcal{L}_2 Z^{-1} \oplus \mathcal{U}_2 = \mathcal{L}_2 Z^{-1} \oplus \mathcal{D}_2 \oplus \mathcal{U}_2 Z .$$

As $\mathcal{X}_2$ is a Hilbert space, we can define $\mathbf{P}_{\mathcal{H}}$ as the projection operator of $\mathcal{X}_2$ onto a subspace $\mathcal{H}$. Often-used projections are $\mathbf{P}$: the projection operator of $\mathcal{X}_2$ onto $\mathcal{U}_2$, and $\mathbf{P}_0$: the projection operator of $\mathcal{X}_2$ onto $\mathcal{D}_2$. With $\mathbf{P}_0$, it is possible to write an operator X in $\mathcal{X}_2$ in terms of its diagonals $X_{[k]} \in \mathcal{D}_2$:

$$X = \sum_{-\infty}^{\infty} Z^k X_{[k]} \,, \qquad X_{[k]} = \mathbf{P}_0(Z^{-k}X) .$$

Realizations

A linear causal time-varying state realization is given by a collection $\{A_k, B_k, C_k, D_k\}_{-\infty}^{\infty}$ of matrices, and consists of the recursion

$$\begin{aligned} x_{i+1} &= x_i A_i + u_i B_i \\ y_i &= x_i C_i + u_i D_i \,, \end{aligned}$$

where u_i is the i-th entry of a sequence $u \in \ell_2$, and likewise for y and x. We allow the dimensions of all matrices to be time-varying. To avoid an abundance of time-indices, and to use signal collections in $\mathcal{X}_2$ as well, we collect the state matrices into diagonals:

$$A = \begin{bmatrix} \ddots & & & \\ & A_0 & & \\ & & A_1 & \\ & & & \ddots \end{bmatrix},$$

and likewise for B, C, D, so that the above realization equation can be written as

$$\begin{array}{rcl} x_{[i+1]}^{(-1)} & = & x_{[i]}A + u_{[i]}B \\ y_{[i]} & = & x_{[i]}C + u_{[i]}D \end{array} \qquad \mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \tag{3}$$

where $u_{[i]}$ and $y_{[i]}$ are the i-th diagonal of $u \in \mathcal{X}_2^{\mathcal{M}}$ and $y \in \mathcal{X}_2^{\mathcal{N}}$, and

$$\begin{array}{rclrcl} A & \in & \mathcal{D}(\mathcal{B}, \mathcal{B}^{(-1)}), & C & \in & \mathcal{D}(\mathcal{B}, \mathcal{N}), \\ B & \in & \mathcal{D}(\mathcal{M}, \mathcal{B}^{(-1)}), & D & \in & \mathcal{D}(\mathcal{M}, \mathcal{N}). \end{array} \tag{4}$$

(We will, throughout this paper, always assume that A, B, C, D are indeed *bounded* diagonal operators.) The space sequence $\mathcal{B}$ is called the system order of the realization; if $A_k : d_k \times d_{k+1}$, then $\mathcal{B}_k = \mathbb{C}^{d_k}$. $\mathbf{T}$ is a realization of T if its entries T_{ij} or diagonals $T_{[i]}$ are given by

$$T_{ij} = \begin{cases} 0, & i > j \\ D_i, & i = j \\ B_i A_{i+1} \cdots A_{j-1} C_j, & i < j \end{cases} \quad \Leftrightarrow \quad T_{[i]} = \begin{cases} 0, & i < 0 \\ D, & i = 0 \\ B^{(i)} A^{(i-1)} \cdots A^{(1)} C, & i > 0. \end{cases} \tag{5}$$

Let ℓ_A denote the spectral radius of the operator AZ: $\ell_A = \lim_{n\to\infty} \| (AZ)^n \|^{1/n}$. If $\ell_A < 1$ then the realization is called strictly stable, and $(I - AZ)$ is invertible in $\mathcal{U}$ so that $T = D + BZ(I - AZ)^{-1}C$. The condition $\ell_A < 1$ is the equivalent in the present setting of the condition 'eigenvalues of A are in the unit disc' in the time-invariant setting.

D-invariant subspaces

A subspace $\mathcal{H}$ of $\mathcal{X}_2^{\mathcal{M}}$ is called (left) D-invariant if $D\mathcal{H} \subset \mathcal{H}$ for all $D \in \mathcal{D}$, and shift-invariant if $Z\mathcal{H} \subset \mathcal{H}$. (All subspaces that we will use are typically D-invariant.) By taking D equal to $D = \operatorname{diag}[\cdots, 0, I, 0, \cdots]$, it is seen that a D-invariant subspace $\mathcal{H}$ falls apart into 'slices' (rows) $\mathcal{H}_i \subset \ell_2^{\mathcal{M}}$, which are subspaces of $\ell_2^{\mathcal{M}}$, and such that

$$\mathcal{H} = \cdots \times \mathcal{H}_0 \times \mathcal{H}_1 \times \cdots.$$

The dimension of $\mathcal{H}$ can then be specified by the sequence of dimensions of each $\mathcal{H}_i$, i.e., the index sequence

$$\text{s-dim}\,\mathcal{H} = [\cdots \ \dim(\mathcal{H}_0) \ \ \dim(\mathcal{H}_1) \ \cdots].$$

If all the $\dim(\mathcal{H}_i)$ are less than some upper bound, then we call $\mathcal{H}$ *locally finite*. In this case, it has a basis representation $\mathbf{F}$ such that

$$\mathcal{H} = \mathcal{D}_2^{\mathcal{B}} \mathbf{F},$$

where $\mathcal{B} = \mathbb{C}^{\text{s-dim}\,\mathcal{H}}$. $\mathbf{F}$ can be obtained by stacking basis vectors of all the $\mathcal{H}_i$ on top of each other. The diagonal operator $\Lambda_{\mathbf{F}} := \mathbf{P}_0(\mathbf{F}\mathbf{F}^*)$ plays the role of Gram operator. If $\Lambda_{\mathbf{F}} = I$, then $\mathbf{F}$ is an orthonormal basis representation. If $\Lambda_{\mathbf{F}}$ is uniformly positive: $\Lambda \gg 0$, then the basis representation is called *strong* (it is a Riesz basis) and can be used to define a projection onto $\mathcal{H}$.

While all of this may seem abstract, the gist is that we will encounter in the paper basis representations that take on familiar forms such as $\mathcal{D}_2(I-AZ)^{-1}C$, for which the Gramian has the notion of observability operator.

Input/output state spaces

Let $T \in \mathcal{U}(\mathcal{M},\mathcal{N})$ be a (causal) transfer operator. When T is viewed as an operator from $\mathcal{X}_2^{\mathcal{M}}$ to $\mathcal{X}_2^{\mathcal{N}}$, then because $\mathcal{X}_2 = \mathcal{L}_2 Z^{-1} \oplus \mathcal{U}_2$, its action on $\mathcal{L}_2 Z^{-1}$ can be decomposed into two operators H_T and K_T:

$$\cdot\, T|_{\mathcal{L}_2 Z^{-1}} = K_T + H_T : \qquad \cdot\, H_T = \mathbf{P}(\,\cdot\, T|_{\mathcal{L}_2 Z^{-1}})\,; \quad \cdot\, K_T = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\cdot\, T|_{\mathcal{L}_2 Z^{-1}})\,.$$

H_T is called the Hankel operator of T. The range and kernel of H_T and H_T^* are D-invariant subspaces with important system-theoretic properties [19]:

$$\begin{array}{lclcl}
\mathcal{K}(T) & = & \ker(\,\cdot\, H_T) & = & \{U \in \mathcal{L}_2 Z^{-1} : \mathbf{P}(UT) = 0\} \\
\mathcal{H}(T) & = & \operatorname{ran}(\,\cdot\, H_T^*) & = & \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 T^*) \\
\mathcal{H}_0(T) & = & \operatorname{ran}(\,\cdot\, H_T) & = & \mathbf{P}(\mathcal{L}_2 Z^{-1} T) \\
\mathcal{K}_0(T) & = & \ker(\,\cdot\, H_T^*) & = & \{Y \in \mathcal{U}_2 : \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(YT^*) = 0\}\,.
\end{array}$$

Note that $\mathcal{H}(T)$ and $\mathcal{H}_0(T)$ are not necessarily closed, although they are closed if T is inner or an isometry. These subspaces provide decompositions of $\mathcal{L}_2 Z^{-1}$ and $\mathcal{U}_2$ as

$$\begin{array}{lclcl}
\overline{\mathcal{H}} & \oplus & \mathcal{K} & = & \mathcal{L}_2 Z^{-1} \\
\overline{\mathcal{H}_0} & \oplus & \mathcal{K}_0 & = & \mathcal{U}_2\,,
\end{array}$$

(the overbar denotes closure). $\mathcal{H}(T)$ is called the (natural) input state space, and $\mathcal{H}_0(T)$ the (natural) output state space of T. If these subspaces are locally finite, then they have the same s-dimension:

$$\text{s-dim}\,\mathcal{H}(T) = \text{s-dim}\,\mathcal{H}_0(T)\,,$$

and T is said to be *locally finite*. The input and output state spaces play a central role in realization theory, as is demonstrated by the following connection.

Let $\{A, B, C, D\}$ be a realization of T. If $\ell_A < 1$, then the operators $\mathbf{F}$ and $\mathbf{F}_0$ defined by

$$\begin{array}{lcl}
\mathbf{F} & := & \left(BZ + BZ\,AZ + BZ\,(AZ)^2 + \cdots \right)^* \\
\mathbf{F}_0 & := & C + AZ\,C + (AZ)^2\,C + \cdots
\end{array} \tag{6}$$

are bounded operators in $\mathcal{L}Z^{-1}$ and $\mathcal{U}$, respectively, and given by $\mathbf{F} = \left(BZ(I-AZ)^{-1}\right)^*$ and $\mathbf{F}_0 = (I-AZ)^{-1}C$, respectively. (In case $\ell_A \le 1$, then $\mathbf{F}$ and $\mathbf{F}_0$ are bounded operators on $\mathcal{D}_2$, and can be defined via (6) on a dense subset of $\mathcal{X}_2$.) We will call $\mathbf{F}$ the controllability operator of the realization, and $\mathbf{F}_0$ the observability operator. The realization is called *[uniformly] controllable* if the controllability Gramian $\Lambda_{\mathbf{F}} := \mathbf{P}_0(\mathbf{F}\mathbf{F}^*)$ is [uniformly] positive, *[uniformly] observable* if the observability Gramian $\Lambda_{\mathbf{F}_0} := \mathbf{P}_0(\mathbf{F}_0\mathbf{F}_0^*)$ is [uniformly] positive, and minimal if it is both controllable and observable. Equivalently, a realization is controllable iff $\ker(\cdot\,\mathbf{F}) = \varnothing$ (i.e., , $\cdot\,\mathbf{F}$ is 'one-to-one') and observable iff $\ker(\cdot\,\mathbf{F}_0) = \varnothing$, whereas the realization is uniformly controllable if $\mathbf{P}_0(\cdot\,\mathbf{F}^*) = \mathcal{D}_2$ (the operator $\mathbf{P}_0(\cdot\,\mathbf{F}^*)$ is 'onto') and uniformly controllable if $\mathbf{P}_0(\cdot\,\mathbf{F}_0^*) = \mathcal{D}_2$.

It can be shown [18] that if $\{A, B, C, D\}$ is a realization of T, then the Hankel operator H_T has a factorization $H_T = \mathbf{P}_0(\cdot\,\mathbf{F}^*)\,\mathbf{F}_0$, where $\mathbf{F}$ and $\mathbf{F}_0$ are as in (6). This factorization shows that, even if an operator $T \in \mathcal{U}$ is locally finite, it might not be possible to have a realization which is both uniformly controllable and uniformly observable [18]. This happens when the range of H_T is not closed, or equivalently, when the range of H_T^* is not closed. It also follows from the factorization of H_T that if a realization is uniformly controllable, then $\mathbf{F}_0$ is a basis representation of $\mathcal{H}_0$:

$$\mathcal{H}_0 \;=\; \mathcal{D}_2\mathbf{F}_0 \;=\; \mathcal{D}_2(I - AZ)^{-1}C$$

and if a realization is uniformly observable,

$$\mathcal{H} \;=\; \mathcal{D}_2\mathbf{F} \;=\; \mathcal{D}_2(I - AZ)^{-*}Z^*B^*\,.$$

If the realization is only controllable or observable, then closures must be taken: $\overline{\mathcal{H}_0} = \overline{\mathcal{D}_2\mathbf{F}_0}$ and $\overline{\mathcal{H}} = \overline{\mathcal{D}_2\mathbf{F}}$, respectively. Finally, it is straightforward to prove, using the relation $\mathbf{F}^* = BZ + \mathbf{F}^*AZ$, that the controllability Gramian of a realization satisfies the Lyapunov equation

$$\Lambda_{\mathbf{F}}^{(-1)} \;=\; A^*\Lambda_{\mathbf{F}}A \;+\; B^*B\,. \tag{7}$$

If $\ell_A < 1$, then the above equation has a unique solution [2] given by

$$\Lambda_{\mathbf{F}} \;=\; \sum_{i=1}^{\infty} (A^{\{i-1\}})^*(B^*B)^{(i)}A^{\{i-1\}}\,,$$

where $A^{\{i\}} := A^{(i)}A^{(i-1)}\cdots A^{(1)}$. Because $\ell_A < 1$, the summation converges strongly. The equation for $\Lambda_{\mathbf{F}}$ in equation (7) is actually a recursion. This is seen when we take the k-th entry of each diagonal in the equation:

$$(\Lambda_{\mathbf{F}})_{k+1} \;=\; A_k^*(\Lambda_{\mathbf{F}})_kA_k + B_k^*B_k\,.$$

Again, since $\ell_A < 1$, the recursion is convergent.

Partial fractions

To end this section, we derive a formula to compute a realization of the upper part of the operator T^*T, when a realization of $T \in \mathcal{U}$ is given. The derivation involves a partial fraction representation which we state in the following lemma.

Lemma 1. *Let $A, F, M \in \mathcal{D}$, and $\ell_A < 1$, $\ell_F < 1$. Then*

$$(I - Z^*F^*)^{-1}Z^*MZ(I - AZ)^{-1} \;=\; (I - Z^*F^*)^{-1}Z^*X \;+\; Y(I - AZ)^{-1}\,, \tag{8}$$

where $X, Y \in \mathcal{D}$ are given by $\begin{cases} X &= F^*Y \\ Y^{(-1)} &= F^*YA + M\,. \end{cases}$

PROOF Upon multiplying (8) by $(I-Z^*F^*)$ at the left and by $(I-AZ)$ at the right, we obtain

$$\begin{array}{rl} Z^*MZ &= Z^*X(I-AZ)+(I-Z^*F^*)Y \\ \Rightarrow & \left\{\begin{array}{lcl} M &=& -XA+Y^{(-1)} \\ 0 &=& X-F^*Y \end{array}\right. \\ \Leftrightarrow & \left\{\begin{array}{lcl} X &=& F^*Y \\ Y^{(-1)} &=& F^*YA+M\,. \end{array}\right. \end{array}$$

□

Lemma 2. *Let $T \in \mathcal{U}$ be given by a state realization $\{A,B,C,D\}$ as $T = D+BZ(I-AZ)^{-1}C$, where $\ell_A < 1$. Then a state realization of the upper part of T^*T is*

$$\left[\begin{array}{cc} A & C \\ D^*B+C^*\Lambda A & D^*D+C^*\Lambda C \end{array}\right]$$

where $\Lambda \in \mathcal{D}$ is the (unique) operator satisfying the Lyapunov equation $\Lambda^{(-1)} = A^*\Lambda A+B^*B$.

PROOF Evaluation of T^*T gives

$$\begin{array}{rcl} T^*T &=& [D^*+C^*(I-Z^*A^*)^{-1}Z^*B^*]\;[D+BZ(I-AZ)^{-1}C] \\ &=& D^*D + C^*(I-Z^*A^*)^{-1}Z^*B^*D + D^*BZ(I-AZ)^{-1}C + \\ && + C^*(I-Z^*A^*)^{-1}Z^*B^*BZ(I-AZ)^{-1}C\,. \end{array}$$

According to lemma 1, the expression $(I-Z^*A^*)^{-1}Z^*B^*BZ(I-AZ)^{-1}$ evaluates as

$$(I-Z^*A^*)^{-1}Z^*B^*BZ(I-AZ)^{-1} = (I-Z^*A^*)^{-1}Z^*X + \Lambda(I-AZ)^{-1}$$

where $X = A^*\Lambda$, and $\Lambda^{(-1)} = A^*\Lambda A+B^*B$. Λ is unique if $\ell_A < 1$, and

$$T^*T = [D^*D+C^*\Lambda C] + [D^*B+C^*\Lambda A]\,Z(I-AZ)^{-1}C + C^*(I-Z^*A^*)^{-1}Z^*\,[A^*\Lambda C+B^*D]\,.$$

□

3. SPECTRAL FACTORIZATION

An operator $U \in \mathcal{X}(\mathcal{M},\mathcal{N})$ is called an isometry if $U^*U = I$, a co-isometry if $UU^* = I$, unitary if it is both isometric and co-isometric, and inner if it is unitary and upper. An operator $W_o \in \mathcal{U}(\mathcal{M},\mathcal{N})$ is defined to be (left) *outer* if

$$\overline{\mathcal{U}_2^{\mathcal{M}}W_o} = \mathcal{U}_2^{\mathcal{N}}\,.$$

W_o is (right) outer if

$$\overline{\mathcal{L}_2Z^{-1}W_o^*} = \mathcal{L}_2Z^{-1}\,.$$

Arveson [8] has shown, in the general context of nest algebras which also applies to our model of time-varying systems, that if $\Omega \in \mathcal{X}$ is a positive operator, there exists an operator $W \in \mathcal{U}$ such that

$$\Omega = W^*W\,.$$

W can be chosen to be outer, in which case the factorization is called a spectral factorization. Related to this fact is another theorem of Arveson in the same paper, which claims that operators in a Hilbert space have an inner-outer factorization

$$W = UW_o$$

where U is an isometry ($U^*U = I$) and W_o is (right) outer.[1] Moreover, if Ω is uniformly positive definite, then Ω has a factorization $\Omega = W_o^* W_o$ where W_o is (left and right) outer and invertible, and hence $\mathcal{L}_2 Z^{-1} W_o^* = \mathcal{L}_2 Z^{-1}$ (no closure is needed) and $W_o^{-1} \in \mathcal{U}$. Any other factor W can be written as $W = UW_o$, where U is now invertible and hence inner.

In this section, we derive an algorithm to compute a time-varying spectral factorization of operators with a state space realization. The computation amounts to the (recursive) solution of a Riccati equation. Such equations have in general a collection of solutions. We show that in order to obtain an outer spectral factor, one must select the smallest positive solution.

The outer factor in a spectral factorization of a positive operator has certain characteristic properties of its input and output state spaces. This is formulated in proposition 5. The recursive version of these properties then produces a Riccati recursive equation, and the existence of the outer factor implies the existence of a (positive) solution to this equation. Other properties of the equation can be derived from the link with outer factors as well.

Lemma 3. *Let $T \in \mathcal{U}(\mathcal{M},\mathcal{M})$ be an outer invertible operator, with state realization $\mathbf{T} = \{A, B, C, D\}$. Then $S = T^{-1} \in \mathcal{U}(\mathcal{M},\mathcal{M})$ has a state realization given by*

$$\mathbf{S} = \begin{bmatrix} A - CD^{-1}B & -CD^{-1} \\ D^{-1}B & D^{-1} \end{bmatrix} \tag{9}$$

PROOF Since T is outer and invertible, $T^{-1} \in \mathcal{U}$, so $S = T^{-1}$ has a realization which is causal. Let $y = uT$, where $u, y \in \mathcal{X}_2^{\mathcal{M}}$. Then $u = yS$, and

$$\left\{ \begin{array}{lcl} x_{[k+1]}^{(-1)} & = & x_{[k]}A + u_{[k]}B \\ y_{[k]} & = & x_{[k]}C + u_{[k]}D \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{lcll} x_{[k+1]}^{(-1)} & = & x_{[k]}(A - CD^{-1}B) & + \; y_{[k]}D^{-1}B \\ u_{[k]} & = & -x_{[k]}CD^{-1} & + \; y_{[k]}D^{-1} \end{array} \right.$$

so that S has a state realization as in (9). □

Lemma 4. *Let $W \in \mathcal{U}$ be boundedly invertible (in $\mathcal{X}$), with inner-outer factorization $W = UW_o$, and suppose that W and W_o have strictly stable realizations with the same (A, C): $W = D + BZ(I - AZ)^{-1}C$, $W_o = D_o + B_o Z(I - AZ)^{-1}C$, $\ell_A < 1$. Let Λ and Λ_o be the controllability Gramians of W and W_o, respectively. Then*

$$\Lambda \geq \Lambda_o\,, \qquad \Lambda = \Lambda_o \quad \textit{iff} \quad U \in \mathcal{D}\,.$$

PROOF Since W_o is outer, a realization of $W_o^{-1} \in \mathcal{U}$ is given by

$$W_o^{-1} = D_o^{-1} - D_o^{-1}BZ(I - A^{\times}Z)^{-1}C\,, \qquad A^{\times} = A - CD^{-1}B\,,$$

[1] Actually, Arveson uses a slightly different definition of outerness (not requiring $\ker(\,\cdot\, W_o)\big|_{\mathcal{L}_2 Z^{-1}} = \varnothing$), so that U can be chosen inner. The resulting inner-outer factorizations are the same when W is invertible. See [17].

so that $U = WW_o^{-1}$ has main diagonal $\mathbf{P}_0(U) = DD_o^{-1}$. Since $U^*U = I$, this implies that $D_o^{-*}D^*DD_o^{-1} \leq I$. Using $W^*W = W_o^*W_o$ and evaluating each term by means of lemma 2 yields the equalities

$$\begin{aligned} D^*D + C^*\Lambda C &= D_o^*D_o + C^*\Lambda_o C \\ D^*B + C^*\Lambda A &= D_o^*B_o + C^*\Lambda_o A \end{aligned}$$

where the controllability Gramians Λ and Λ_o are specified (uniquely) by

$$\begin{aligned} \Lambda^{(-1)} &= A^*\Lambda A + B^*B \\ \Lambda_o^{(-1)} &= A^*\Lambda_o A + B_o^*B_o \end{aligned}$$

The first equation is equivalent to

$$D_o^{-*}C^*(\Lambda - \Lambda_o)CD_o^{-1} = I - D_o^{-*}D^*DD_o^{-1}$$

and since $D_o^{-*}D^*DD_o^{-1} \leq I$, it follows that $\Lambda - \Lambda_o \leq 1$.

The proof that $\Lambda = \Lambda_o \;\Rightarrow U \in \mathcal{D}$ is also a straightforward consequence of these equations. □

Recall the definitions of input and output state spaces of T as $\mathcal{H}(T) = \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\mathcal{U}_2T^*)$, $\mathcal{H}_0(T) = \mathbf{P}(\mathcal{L}_2Z^{-1}T)$.

Proposition 5. *Let $T \in \mathcal{U}(\mathcal{M},\mathcal{M})$ be such that $T^* + T \gg 0$. In addition, let $W \in \mathcal{U}(\mathcal{M},\mathcal{M})$ be an invertible factor of $T^* + T = W^*W$. Then $\mathcal{H}_0(T) \subset \mathcal{H}_0(W)$. If W is outer, $\mathcal{H}_0(T) = \mathcal{H}_0(W)$. In particular, there exists a realization of an outer W that has the same (A,C) pair as a realization of T.*

PROOF According to Arveson [8], there exists an invertible operator $W \in \mathcal{U}$ such that

$$T^* + T \;=\; W^*W\,.$$

In general, $\mathcal{L}_2Z^{-1}W^* \subset \mathcal{L}_2Z^{-1}$, and $\mathcal{L}_2Z^{-1}W^* = \mathcal{L}_2Z^{-1}$ if and only if W is outer. Thus

$$\begin{aligned} \mathcal{H}_0(T) &= \mathbf{P}(\mathcal{L}_2Z^{-1}T) \\ &= \mathbf{P}(\mathcal{L}_2Z^{-1}[T + T^*]) && [\text{since } T^* \in \mathcal{L}] \\ &= \mathbf{P}(\mathcal{L}_2Z^{-1}W^*W) \\ &\subset \mathbf{P}(\mathcal{L}_2Z^{-1}W) \;=\; \mathcal{H}_0(W)\,. \end{aligned}$$

If W is outer, then $\mathcal{L}_2Z^{-1}W^* = \mathcal{L}_2Z^{-1}$ and the inclusion in the above derivation becomes an identity: W outer $\;\Rightarrow \mathcal{H}_0(T) = \mathcal{H}_0(W)$. If $\{A,B,C,D\}$ is a realization of T with $\ell_A < 1$, then $\mathcal{H}_0(T) = \mathcal{D}_2(I - AZ)^{-1}C$ (if the realization of T is uniformly controllable) or, more generally, $\mathcal{H}_0(T) \subset \mathcal{D}_2(I - AZ)^{-1}C$. Hence, it is clear that a realization of an outer W can have the same (A,C)-pair as a realization of T. □

Note that not necessarily $\mathcal{H}_0(T) = \mathcal{H}_0(W) \Rightarrow W$ outer , as a simple time-invariant example shows. The proposition, along with lemma 4, assures that a minimal degree factor W of $T + T^* \gg 0$ is obtained by taking a realization of W to have the (A,C)-pair as a realization of T, and that this factor is outer if the controllability Gramian of this realization is as small as possible. This observation forms the main part of the proof of the following theorem, which can be used to actually compute the realization of the outer factor if a realization of T is given.

Theorem 6. *Let $T \in \mathcal{U}(\mathcal{M},\mathcal{M})$ be a locally finite operator with an observable state realization $\{A,B,C,D\}$ such that $\ell_A < 1$. Then $T^* + T \gg 0$ if and only if a solution $\Lambda \in \mathcal{D}$, $\Lambda \geq 0$ exists of*

$$\Lambda^{(-1)} = A^*\Lambda A + \left[B^* - A^*\Lambda C\right](D + D^* - C^*\Lambda C)^{-1}\left[B - C^*\Lambda A\right] \tag{10}$$

such that $D + D^* - C^*\Lambda C \gg 0$.

If $T^* + T \gg 0$, let $W \in \mathcal{U}(\mathcal{M},\mathcal{M})$ be an invertible factor of $T^* + T = W^*W$. A realization $\{A, B_W, C, D_W\}$ for W such that W is outer is then given by the smallest solution $\Lambda \geq 0$, and

$$\begin{cases} D_W = (D + D^* - C^*\Lambda C)^{1/2} \\ B_W = D_W^{-*}\left[B - C^*\Lambda A\right] . \end{cases}$$

The realization of W is observable and [uniformly] controllable, if T is so.

PROOF Let the realization of T satisfy the given requirements, and suppose that $T + T^* \gg 0$. Then $T + T^* = W^*W$, where W is outer. According to proposition 5, W can have the same (A,C) pair as T. Hence assume that $W = D_W + B_W Z(I - AZ)^{-1}C$, and denote its controllability Gramian by Λ. Then, with help of lemma 2, this realization satisfies

$$\begin{aligned} D + D^* &= D_W^* D_W + C^*\Lambda C\,, \qquad D_W^* D_W \gg 0 \\ BZ(I - AZ)^{-1}C &= \left[D_W^* B_W + C^*\Lambda A\right] Z(I - AZ)^{-1}C \\ \Lambda^{(-1)} &= A^*\Lambda A + B_W^* B_W\,, \qquad \Lambda \geq 0\,. \end{aligned}$$

Because the realization of T is observable, the operator $\cdot\,(I - AZ)^{-1}C$ is one-to-one by definition, and the above set of equations reduce to

$$\begin{aligned} D + D^* &= D_W^* D_W + C^*\Lambda C\,, \qquad D_W^* D_W \gg 0 \\ B &= D_W^* B_W + C^*\Lambda A \\ \Lambda^{(-1)} &= A^*\Lambda A + B_W^* B_W\,, \qquad \Lambda \geq 0\,. \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} D_W &= (D + D^* - C^*\Lambda C)^{1/2} \\ B_W &= D_W^{-*}\left[B - C^*\Lambda A\right] \\ \Lambda^{(-1)} &= A^*\Lambda A + \left[B^* - A^*\Lambda C\right](D + D^* - C^*\Lambda C)^{-1}\left[B - C^*\Lambda A\right]\,, \end{aligned}$$

(D_W, and hence B_W, are determined up to a left diagonal unitary factor), so that Λ satisfies the given Riccati equation. In fact, we showed that if $T + T^* \gg 0$, the existence of an outer factor implies that there is a solution Λ of the Riccati equation which is positive semi-definite, and such that also $D + D^* - C^*\Lambda C \gg 0$. The converse, to show that $T + T^* \gg 0$ if these quantities are positive semi-definite, resp. uniformly positive, is obvious at this point, since such a solution directly specifies a realization of an invertible factor W of $T + T^*$. If this solution Λ is the smallest possible solution, then, by lemma 4, W is outer. □

The above theorem can be extended to realizations without observability constraint.

Theorem 7. *Theorem 6 holds also if the realization of T is not observable.*

The proof of this theorem is technical and given in appendix A.

Theorems 6 and 7 can also be specified in two alternate forms, familiar from the time-invariant context [7, 11]:

Corollary 8. **(The time-varying positive real lemma)** *Let $T \in \mathcal{U}$ be a locally finite operator with state realization $\{A, B, C, D\}$ such that $\ell_A < 1$.*

Then $T^* + T \gg 0$ if and only if there exist diagonal operators Λ, Q, B'_W with $\Lambda \geq 0$ and $Q \gg 0$ satisfying

$$\begin{aligned} \Lambda^{(-1)} &= A^*\Lambda A + B'^*_W Q B'_W \\ B'^*_W Q &= B^* - A^*\Lambda C \\ Q &= D + D^* - C^*\Lambda C\,. \end{aligned}$$

PROOF In view of theorems 6 and 7, it suffices to make the connection $Q = D^*_W D_W$ and $B_W = D_W B'_W$. □

Corollary 9. **(Time-varying spectral factorization)** *Let $\Omega \in \mathcal{X}$ be a Hermitian operator whose upper part is locally finite with state realization $\{A, B, C, D\}$ satisfying $\ell_A < 1$, i.e.,*

$$\Omega \;=\; D \,+\, BZ(I - AZ)^{-1}C \,+\, C^*(I - Z^*A^*)^{-1}Z^*B^*\,,$$

Then $\Omega \gg 0$ if and only if there exists a solution $\Lambda \in \mathcal{D}, \Lambda \geq 0$ of

$$\Lambda^{(-1)} \;=\; A^*\Lambda A \,+\, \left[B^* - A^*\Lambda C\right](D - C^*\Lambda C)^{-1}\left[B - C^*\Lambda A\right]\,, \tag{11}$$

such that $D - C^*\Lambda C \gg 0$.

If $\Omega \gg 0$ and Λ is the smallest positive solution, then a realization $\{A, B_W, C, D_W\}$ for an outer factor W of Ω is given by

$$\begin{aligned} D_W &= (D \,-\, C^*\Lambda C)^{1/2} \\ B_W &= D_W^{-*}\left[B - C^*\Lambda A\right]\,. \end{aligned}$$

If the realization $\{A, B, C, D\}$ is observable and controllable resp. uniformly controllable, then $\Lambda > 0$ resp. $\Lambda \gg 0$: the realization for W is observable and [uniformly] controllable.

4. COMPUTATIONAL ISSUES

In this section, we consider some computational issues that play a role in actually computing a spectral factorization of a positive operator Ω with a locally finite realization given as in (11). First, note that taking the k-th entry along the diagonal of each operator in (11) leads to the Riccati *recursion*

$$\Lambda_{k+1} \;=\; A_k^*\Lambda_k A_k \,+\, \left[B_k^* - A_k^*\Lambda_k C_k\right](D_k - C_k^*\Lambda_k C_k)^{-1}\left[B_k - C_k^*\Lambda_k A_k\right]\,, \tag{12}$$

and with Λ_k known, $(B_W)_k$, $(D_W)_k$ also follow locally:

$$\begin{aligned} (D_W)_k &= (D_k \,-\, C_k^*\Lambda_k C_k)^{1/2} \\ (B_W)_k &= (D_W^{-*})_k\left[B_k - C_k^*\Lambda_k A_k\right] \end{aligned}$$

Hence all that is needed in practical computations is an initial point for the recursion of Λ_k. We consider some special cases for which such an initial point can indeed be obtained.

One general observation is that, since there may be more than one positive solution Λ, there also may be more than one initial point Λ_k. Outer factors are obtained by choosing the smallest positive solution, which implies taking the smallest positive initial point: since $\Lambda \le \Lambda' \Rightarrow \Lambda_k \le \Lambda'_k (\forall k)$, a single Λ_k is part of the smallest solution if and only if the corresponding Λ is the smallest.

Finite matrices

One case in which exact initial conditions can be obtained is the case where $\Omega \in \mathcal{X}(\mathcal{M}, \mathcal{M})$ is actually a finite matrix, i.e., where

$$\mathcal{M} \;=\; \cdots \varnothing \times \mathcal{M}_0 \times \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \times \varnothing \times \cdots ,$$

In this case, Ω is a finite $n \times n$ (block) matrix, and a realization for Ω can start off with 0 states at point 1 in time. Since the dimension of Λ follows that of A, an exact initial point for the recursion is $\Lambda_1 = [\,\cdot\,]$ (a 0×0 matrix), which is also the smallest positive initial point. The spectral factorization reduces for finite matrices to a Cholesky factorization, and the resulting algorithm is an efficient way to compute Cholesky factorizations for (large) matrices with a sparse state space. Other, related, results on computational linear algebra in a state space context can be found in [22, 18].

Initial time-invariance

A second class of systems are systems which are time-invariant before some point in time, say $k = 1$. Then, before point $k = 1$, all Λ_k are equal to each other, and in particular $\Lambda_0 = \Lambda_1$. Hence the recursion for Λ reduces to an algebraic equation

$$\Lambda_0 \;=\; A_0^* \Lambda_0 A_0 \;+\; \left[B_0^* - A_0^* \Lambda C_0\right] (D_0 - C_0^* \Lambda_0 C_0)^{-1} \left[B_0 - C_0^* \Lambda_0 A_0\right] \;,$$

which is the classical time-invariant Riccati equation. A solution to this equation can be obtained in one of the classical ways, e.g., as the solution of a Hamiltonian equation. Multiple solutions exist, and in order to obtain an outer spectral factor W, the smallest positive solution of the above equation must be chosen. Because the Λ_k $(k > 0)$ are determined by Λ_0 via the recursion (12), the resulting Λ will also be the smallest solution for all time.

Periodic systems

If Ω is periodically time-varying, with period n say, then one can apply the usual time-invariance transformation, by considering a block system consisting of n consecutive state realization sections. Since the block-system is time-invariant, one can compute the smallest solution Λ_1 from the resulting block-Riccati equation with the classical techniques, and Λ_1 is an exact initial condition to compute the realization of the spectral factor for time points $2, \cdots, n$. As usual, such a technique may not be attractive if the period is large.

Unknown initial conditions

Finally, we consider the more general case where Ω is not completely specified, but only, say, its 'future' submatrix $[\Omega_{i,j}]_0^\infty$ is known. The unknown 'past' of Ω is assumed to be such that $\Omega \gg 0$. In this case, the exact initial point for the recursion of Λ_k is unknown. It is possible to start the recursion (12) from an approximate initial point, for which typically $\hat{\Lambda}_0 = 0$ is chosen. The convergence of this choice is investigated in appendix B. It is shown in proposition 16 that when the realization $\{A, B, C, D\}$ is observable and has $\ell_A < 1$, then $\hat{\Lambda}_k$ (corresponding to the recursion (12) with initial point $\hat{\Lambda}_0 = 0$) converges to Λ_k, the smallest exact solution obtained with the correct initial point Λ_0.

5. CONNECTIONS

In this section, we point out some of the connections between the spectral factorization results of the preceding sections, and other incarnations of the time-varying Riccati equation, arising in the orthogonal embedding problem and inner-outer factorizations.

Orthogonal embedding

The orthogonal embedding problem is, given a transfer operator T of a causal bounded discrete-time linear system, to extend this system by adding more inputs and outputs to it such that the resulting system Σ,

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

is lossless: $\Sigma^*\Sigma = I$, $\Sigma\Sigma^* = I$, and has T as its partial transfer when the extra inputs are forced to zero: $T = \Sigma_{11}$. This problem is also known as the Darlington problem in classical network theory [10, 7]. Since the losslessness of Σ implies $T^*T + \Sigma_{21}^*\Sigma_{21} = I$, it will be possible to find solutions to the embedding problem only if T is contractive: $I - T^*T \geq 0$. The translation of this condition into a state realization context, leading to a solution of the embedding problem, is given in the following theorem.

Theorem 10. **([21, 20])** *Let $T \in \mathcal{U}$ be a locally finite operator with state realization $\{A, B, C, D\}$ where $\ell_A < 1$. If T is strictly contractive, then there exists $M \in \mathcal{D}$, $M \geq 0$, such that $I - D^*D - C^*MC \gg 0$ and*

$$M^{(-1)} = A^*MA + B^*B + \left[A^*MC + B^*D\right](I - D^*D - C^*MC)^{-1}\left[D^*B + C^*MA\right]. \qquad (13)$$

If in addition the state space realization is uniformly controllable, then $M \gg 0$.

As in the spectral factorization problem, once a solution M of the Riccati equation is obtained, a solution Σ of the embedding problem follows straightforwardly. In this respect, we mention the time-varying bounded real lemma that is connected to the embedding problem:

Theorem 11. **(Time-varying bounded real lemma [20])** *Let $T \in \mathcal{U}(\mathcal{M}_1, \mathcal{N}_1)$ be as in theorem 10, with $\{A, B, C, D\}$ a uniformly controllable realization. Let $A \in \mathcal{D}(\mathcal{B}, \mathcal{B}^{(-1)})$.*

(i) If $\mathbf{T}$ *is uniformly observable, then* T *is contractive if and only if the set of equations*

$$\left\{\begin{array}{lllllll} A^*MA & + & B^*B & + & B_2^*B_2 & = & M^{(-1)} \\ C^*MC & + & D^*D & + & D_{21}^*D_{21} & = & I \\ A^*MC & + & B^*D & + & B_2^*D_{21} & = & 0 \end{array}\right. \tag{14}$$

has a solution $M \in \mathcal{D}(\mathcal{B},\mathcal{B})$, $B_2 \in \mathcal{D}(\mathcal{N}_1,\mathcal{B}^{(-1)})$, $D_{21} \in \mathcal{D}(\mathcal{N}_1,\mathcal{N}_1)$ *such that* $M \gg 0$.

(ii) If T *is strictly contractive, then there exists* $M \gg 0$ *and operators* B_2, D_{21} *that satisfy the equations (14).*

Exact initial points for the recursion for M corresponding to (13) can again be obtained in special cases of general interest, and it can be proven that the solution $\hat{M}_k$, obtained by the approximate initial point $\hat{M}_0 = 0$, will converge to the exact solution M_k. With M solved from the Riccati recursion and B_2, D_{21} determined using (14), a realization $\mathbf{\Sigma}$ of the orthogonal embedding Σ follows as

$$\mathbf{\Sigma}_i = \begin{bmatrix} R_i & & \\ & I & \\ & & I \end{bmatrix} \left[\begin{array}{cc|c} A_i & C_i & C_{2,i} \\ B_i & D_i & D_{12,i} \\ \hline B_{2,i} & D_{21,i} & D_{22,i} \end{array}\right] \begin{bmatrix} R_{i+1}^{-1} & & \\ & I & \\ & & I \end{bmatrix} \tag{15}$$

where the state transformation R_i is a factor of M_i such that $M_i = R_i^* R_i$, and $C_{2,i}$, $D_{12,i}$ and $D_{21,i}$ form a unitary extension such that $\mathbf{\Sigma}_i$ is a square and unitary matrix. $\{A, B_2, C, D_{21}\}$ is a realization for Σ_{21}, satisfying $I - T^*T = \Sigma_{21}^*\Sigma_{21}$. We have thus obtained a spectral factorization of $I - T^*T$. Via this connection, the time-varying bounded real lemma can also be obtained starting from the positive real lemma (theorems 6 and 7). This leads to a variant of the above theorem which is more general, as it also follows that Σ_{21} is in fact an outer operator.

Theorem 12. (Time-varying bounded real lemma derived from positive real lemma) *Let* $T \in \mathcal{U}(\mathcal{M}_1,\mathcal{N}_1)$ *be a locally finite operator with a state realization* $\{A,B,C,D\}$ *such that* $\ell_A < 1$. *Then* $I - T^*T \gg 0$ *if and only if there exists a solution* $M \in \mathcal{D}(\mathcal{B},\mathcal{B})$, $M \geq 0$ *of*

$$M^{(-1)} = A^*MA + B^*B + \left[A^*MC + B^*D\right](I - D^*D - C^*MC)^{-1}\left[D^*B + C^*MA\right] \tag{16}$$

such that $I - D^*D - C^*MC \gg 0$. *If in addition the realization of* T *is observable and [uniformly] controllable, then* M *is [uniformly] positive.*

If $I - T^*T \gg 0$, *let* $W \in \mathcal{U}(\mathcal{N}_1,\mathcal{N}_1)$ *be a factor of* $I - T^*T = W^*W$. *A realization* $\{A, B_W, C, D_W\}$ *for* W *such that* W *is outer is then given by the smallest solution* $M \geq 0$ *of the above equation, and*

$$\left\{\begin{array}{lll} D_W & = & (I - D^*D - C^*MC)^{1/2} \\ B_W & = & -D_W^{-*}\left[D^*B + C^*MA\right] . \end{array}\right. \tag{17}$$

Proof Since $\ell_A < 1$, the Lyapunov equation

$$\Lambda^{(-1)} = A^*\Lambda A + B^*B$$

has a unique solution $\Lambda \geq 0$. By lemma 2, an expression for $I - T^*T$ is

$$I - T^*T = (I - D^*D - C^*\Lambda C) - \left[D^*B + C^*\Lambda A\right] Z(I - AZ)^{-1}C - C^*(I - Z^*A^*)^{-1}Z^*\left[B^*D + A^*\Lambda C\right] .$$

The implied realization for the upper part of $I - T^*T$ need not be controllable. Theorem 7 claims that $I - T^*T \gg 0$ if and only if there exists a solution $P \in \mathcal{D}$ of

$$P^{(-1)} = A^*PA + \left[B^*D + A^*(\Lambda + P)C\right] (I - D^*D - C^*(\Lambda + P)C)^{-1} \left[D^*B + C^*(\Lambda + P)A\right]$$

such that $I - D^*D - C^*(\Lambda + P)C \gg 0$ and $P \geq 0$. As a consequence, the operator $M = \Lambda + P$ is positive semi-definite and satisfies equation (16). If the realization of T is observable and [uniformly] controllable, then $\Lambda > 0$ [$\Lambda \gg 0$], and the same holds for M.

Theorem 7 in addition shows that the realization $\{A, B_W, C, D_W\}$, with D_W, B_W as given in (17), defines an outer factor W of $I - T^*T = W^*W$ if M is the smallest positive semi-definite solution. □

Inner-outer factorization

A realization of the outer factor in an inner-outer factorization can also be computed via a Riccati equation:

Theorem 13. **([17])** *Let $T \in \mathcal{U}$ be a locally finite transfer operator, let $\mathbf{T} = \{A, B, C, D\}$ be an observable realization of T, and assume $\ell_A < 1$. Then a realization of the outer factor T_0 of T, so that $T_0 = U^*T$ is outer and $U^*U = I$, is given by*

$$\mathbf{T}_0 = \begin{bmatrix} I & \\ & R^* \end{bmatrix} \begin{bmatrix} A & C \\ C^*MA + D^*B & C^*MC + D^*D \end{bmatrix}$$

where $M \geq 0$ is the solution of maximal rank of

$$M^{(-1)} = A^*MA + B^*B - \left[A^*MC + B^*D\right] (D^*D + C^*MC)^{\dagger} \left[D^*B + C^*MA\right]$$

and R is a minimal (full range) factor of $RR^* = (D^*D + C^*MC)^{\dagger}$, provided the pseudo-inverse $(\cdot)^{\dagger}$ is bounded.

(The pseudo-inverse is always bounded if the outer factor has closed range, in particular if it is invertible.) Using lemma 2, and assuming $T^*T \gg 0$, one can show that, indeed, $T^*T = T_0^*T_0$, by deriving that the realizations of the upper parts are equal. With lemma 2, the realization of the upper part of $T_0^*T_0$ is

$$\begin{bmatrix} A & C \\ (D^*B + C^*MA) + C^*\Lambda'A & (D^*D + C^*MC) + C^*\Lambda'C \end{bmatrix} \tag{18}$$

where Λ' is the unique operator satisfying the Lyapunov equation

$$\Lambda'^{(-1)} = A^*\Lambda'A + \left[B^*D + A^*MC\right] (D^*D + C^*MC)^{-1} \left[D^*B + C^*MA\right] .$$

Consequently, $(\Lambda' + M)^{(-1)} = A^*(\Lambda' + M)A + B^*B$, so that $\Lambda = \Lambda' + M$ satisfies the Lyapunov equation $\Lambda^{(-1)} = A^*\Lambda A + B^*B$. With Λ, the realization (18) becomes

$$\begin{bmatrix} A & C \\ B^*D + C^*\Lambda A & D^*D + C^*\Lambda C \end{bmatrix} ,$$

which is the same realization as that of the upper part of T^*T in lemma 2. Conversely, one can try to derive theorem 13 from the spectral factorization theorem via this way, for the special case where T^*T is invertible (theorem 13 is more general).

6. CONCLUSIONS

In this paper, we have investigated some of the properties of a Riccati equation with time-varying parameters and time-varying dimensions. By analysis of the underlying fundamental problem, in this case the problem of spectral factorization, it was possible to prove the existence of a positive Hermitian solution and to show how this solution is connected to the outer factor of the factorization. (It is, in fact, the controllability Gramian of a realization of the factor.) The relatively clean derivation of this result was greatly enhanced by the use of a recently developed index-free notation and the corresponding realization theory for time-varying systems. Via the connection with spectral factorization, it was also possible to give a straightforward proof of convergence of the recursion, from an approximate initial point to the exact solution. The convergence property is instrumental in practical applications, e.g., in cases where the state realization of a positive operator is known only on a finite interval, and a spectral factor of this operator should be computed. Such applications will be studied to a larger extend in future publications.

A. Proof of theorem 7

In this appendix, we will give the proof of theorem 7. In order to relax the observability condition, we first mention some properties on realizations of an outer operator and its inverse.

Proposition 14. *Let $T \in \mathcal{U}(\mathcal{M},\mathcal{M})$ be an outer invertible operator, with state realization* $\mathbf{T} = \{A, B, C, D\}$. *Then $S = T^{-1} \in \mathcal{U}(\mathcal{M},\mathcal{M})$ has a state realization given by (9). Moreover,* $\mathbf{T}$ *is [uniformly] controllable if and only if* $\mathbf{S}$ *is [uniformly] controllable,* $\mathbf{T}$ *is [uniformly] observable if and only if* $\mathbf{S}$ *is [uniformly] observable. Let $A^{\times} = A - CD^{-1}B$. If $\ell_A < 1$ and* $\mathbf{T}$ *is controllable or observable, then $\ell_{A^{\times}} < 1$.*

PROOF The model of S was derived in lemma 3). To prove the remaining properties, let

$$\cdot\, T|_{\mathcal{L}_2Z^{-1}} = K_T + H_T : \qquad \cdot\, H_T = \mathbf{P}(\,\cdot\, T|_{\mathcal{L}_2Z^{-1}})\,; \quad \cdot\, K_T = \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\,\cdot\, T|_{\mathcal{L}_2Z^{-1}})\,.$$

The Hankel operator H_T has a factorization in terms of the controllability and observability operators $\mathbf{F}$ and $\mathbf{F}_0$ defined in (6) as $H_T = \mathbf{P}_0(\,\cdot\,\mathbf{F}^*)\,\mathbf{F}_0$. Partition $u \in \mathcal{X}_2^{\mathcal{M}}$ into a past and a future part: $u = u_p + u_f \in \mathcal{L}_2Z^{-1} \oplus \mathcal{U}_2$, and partition y likewise. Then

$$y = uT \quad \Leftrightarrow \quad \begin{cases} y_p &= u_pK_T \\ x_{[0]} &= \mathbf{P}_0(u_p\mathbf{F}^*) \\ y_f &= u_fT + x_{[0]}\mathbf{F}_0 \end{cases}$$

Because T is invertible in $\mathcal{U}$, K_T is invertible, and hence, the above set of equations is equivalent to

$$u = yS \quad \Leftrightarrow \quad \begin{cases} u_p &= y_pK_T^{-1} \\ x_{[0]} &= \mathbf{P}_0(y_pK_T^{-1}\mathbf{F}^*) \\ u_f &= y_fT^{-1} - x_{[0]}\mathbf{F}_0T^{-1} \end{cases}$$

It follows that $\mathbf{S}$ has controllability and observability operators given by

$$\mathbf{F}_S = \mathbf{F}K_T^{-*}\,, \qquad \mathbf{F}_{0,S} = -\mathbf{F}_0T^{-1}\,.$$

These operators inherit the one-to-one and onto properties of the controllability and observability operators of T.

Finally, to show that $\ell_{A^\times} < 1$ if $\ell_A < 1$ and $\Lambda_{\mathbf{F}} > 0$ or $\Lambda_{\mathbf{F}_0} > 0$, we invoke the following extension of a result proven in [18, lemma 3.18]: If $\Lambda_{\mathbf{F}} > 0$, then

$$\mathbf{F} \text{ is bounded on } \mathcal{X}_2 \qquad \Leftrightarrow \qquad \ell_A < 1\,.$$

Applying this result twice yields, if $\Lambda_{\mathbf{F}} > 0$,

$$\ell_A < 1 \quad \Rightarrow \quad \mathbf{F} \text{ bounded on } \mathcal{X}_2 \quad \Rightarrow \quad \mathbf{F}_S \text{ bounded on } \mathcal{X}_2 \quad \Rightarrow \quad \ell_A^\times < 1\,.$$

A similar result holds if $\Lambda_{\mathbf{F}_0} > 0$. □

We are now in a position to prove theorem 7, i.e., theorem 6 without the observability constraint.

PROOF of theorem 7. We will first transform the given realization into one that is observable. Factor the observability Gramian $\Lambda_{\mathbf{F}_0}$ of the given realization as

$$\Lambda_{\mathbf{F}_0} = X^* \begin{bmatrix} \Lambda_{11} & \\ & 0 \end{bmatrix} X\,,$$

where X is an invertible state transformation and $\Lambda_{11} > 0$. Applying X^{-1} as state transformation to $\mathbf{T}$ leads to a realization $\mathbf{T}' = \{A', B', C', D\}$ given by

$$\begin{bmatrix} A' & C' \\ B' & D \end{bmatrix} = \begin{bmatrix} X^{-*} & \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} X^{*(-1)} & \\ & I \end{bmatrix}.$$

Partition A', B', C' conform the partitioning of Λ. It follows that

$$A' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad B' = [\, B_1 \quad B_2 \,]\,, \qquad C' = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}.$$

The subsystem $\{A_{11}, B_1, C_1, D\}$ is an observable realization of T, with $\ell_{A_{11}} < 1$.

Suppose P is a Hermitian solution of

$$P^{(-1)} = A'^* P A' \;+\; \left[B'^* - A'^* P C'\right] (D + D^* - C'^* P C')^{-1} \left[B' - C'^* P A'\right] \tag{19}$$

Partition P conformably to the partitioning of A: $P = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{smallmatrix}\right]$. Then equation (19) is equivalent to the three equations

$$\begin{array}{lll}
(a) \;\; P_{11}^{(-1)} & = & A_{11}^* P_{11} A_{11} \;+\; [B_1^* - A_{11}^* P_{11} C_1]\, (D + D^* - C_1^* P_{11} C_1)^{-1}\, [B_1 - C_1^* P_{11} A_{11}] \\
(b) \;\; P_{12}^{(-1)} & = & (A_{11}^\times)^* P_{12} A_{22} \;+\; A_{11}^* P_{11} A_{12} + [B_1^* - A_{11}^* P_{11} C_1]\, (D + D^* - C_1^* P_{11} C_1)^{-1}\, [B_2 - C_1^* P_{11} A_{12}] \\
(c) \;\; P_{22}^{(-1)} & = & A_{22}^* P_{22} A_{22} \;+\; A_{12}^* P_{11} A_{12} + A_{12}^* P_{12} A_{22} + A_{22}^* P_{12}^* A_{12} + \\
 & & +\;\; [B_2^* - (A_{12}^* P_{11} + A_{22}^* P_{12}^*) C_1]\, (D + D^* - C_1^* P_{11} C_1)^{-1}\, [B_2 - C_1^* (P_{11} A_{12} + P_{12} A_{22})]
\end{array}$$

where $A_{11}^\times := A_{11} - C_1 (D + D^* - C_1^* P_{11} C_1)^{-1} \left[B_1 - C_1^* P_{11} A_{11}\right]$.

According to theorem 6, the first equation has solutions $P_{11} \geq 0$ such that $D + D^* - C_1^* P_{11} C_1 \gg 0$, if and only if $T + T^* \gg 0$. Take P_{11} to be the smallest positive solution, then W is outer and the has an observable realization $\{A_{11}, B_{W1}, C_1, D_W\}$ with D_W and B_{1W} given by

$$\begin{aligned} D_W^* D_W &= D + D^* - C_1^* P_{11} C_1 \\ B_{1W} &= D_W^{-*} \left[B_1 - C_1^* P_{11} A_{11} \right] . \end{aligned}$$

According to proposition 14, W^{-1} has a realization with A-operator given by $A_{11}^\times = A_{11} - C_1 D_W^{-1} B_{1W} = A_{11} - C_1 (D + D^* - C_1^* P_{11} C_1)^{-1} \left[B_1 - C_1^* P_{11} A_{11} \right]$, and satisfying $\ell_{A_{11}^\times} < 1$ (since $\ell_{A_{11}} < 1$ and the realization of W is observable). The second equation is a kind of Lyapunov equation in P_{12}, as only the first term of the right-hand side is dependent on P_{12}. Given P_{11}, it has a unique bounded solution since $\ell_{A_{11}^\times} < 1$ and $\ell_{A_{22}} < 1$. The last equation is a Lyapunov equation in P_{22}, and also has a unique bounded solution.

Also note that $D + D^* - C_1^* P_{11} C_1 = D + D^* - C'^* P C'$. Hence we showed

$$T + T^* \gg 0 \quad \Leftrightarrow \quad \exists\, P \text{ satisfying (19), such that } D + D^* - C'^* P C' \gg 0 .$$

The latter also implies $P \geq 0$. With $\Lambda = X^{-1} P X^{-*}$, Λ is in fact independent of the chosen state transformation X and satisfies the statements of the theorem.

The realization of W can be extended to a non-minimal one that is specified in terms of P as $\{A', B'_W, C', D_W\}$, where the newly introduced quantity B'_W is given by $B'_W = D_W^{-*} \left[B' - C'^* P A' \right] = [B_{1W} \quad B_{2W}]$, for a certain B_{2W}. Upon state-transforming this realization by X, we obtain a realization of W as $\{A, B_W, C, D_W\}$, where D_W is as before, and B_W is specified in terms of Λ as $B_W = B'_W X^{-*(-1)} = D_W^{-*}(B - C^* \Lambda A)$. □

B. Convergence of the Riccati recursion (12)

In this section, we study the convergence of an approximate solution $\hat{\Lambda}_k$ ($k \geq 0$) to the Riccati recursion (12), if the recursion is started with $\hat{\Lambda}_0 = 0$ rather than the exact initial point Λ_0. It is shown that $\hat{\Lambda}_k \to \Lambda_k$ for $k \to \infty$, when $\Omega \gg 0$, $\ell_A < 1$ and the given realization is observable. Similar results are well known for the time-invariant case, and for the time-varying case some results are known from the connection of the Riccati recursion with Kalman filtering (cf. [1, 5]). However, the derivation given below is more general because state dimensions are allowed to vary, and hence A_k cannot be assumed to be square and invertible, as required in [1].

Consider the following block decomposition of the matrix representation of $\Omega = W^* W$, and a related operator $\hat{\Omega} = \hat{W}^* \hat{W}$:

$$\begin{aligned} \Omega &= \left[\begin{array}{c|cc} \underline{\Omega}_{11} & \underline{\Omega}_{12} & \underline{\Omega}_{13} \\ \hline \underline{\Omega}_{12}^* & \underline{\Omega}_{22} & \underline{\Omega}_{23} \\ \underline{\Omega}_{13}^* & \underline{\Omega}_{23}^* & \underline{\Omega}_{33} \end{array} \right], \qquad W = \left[\begin{array}{c|cc} W_{11} & W_{12} & W_{13} \\ \hline & W_{22} & W_{23} \\ & & W_{33} \end{array} \right] \\ \hat{\Omega} &= \left[\begin{array}{c|cc} \underline{\Omega}_{11} & 0 & 0 \\ \hline 0 & \underline{\Omega}_{22} & \underline{\Omega}_{23} \\ 0 & \underline{\Omega}_{23}^* & \underline{\Omega}_{33} \end{array} \right], \qquad \hat{W} = \left[\begin{array}{c|cc} \hat{W}_{11} & 0 & 0 \\ \hline & \hat{W}_{22} & \hat{W}_{23} \\ & & \hat{W}_{33} \end{array} \right] . \end{aligned} \tag{20}$$

In these decompositions,[2] $\underline{\Omega}_{11}$ corresponds to $[\Omega_{i,j}]_{-\infty}^{-1}$, $\underline{\Omega}_{22} = [\Omega_{i,j}]_0^{n-1}$ is a finite $n \times n$ matrix (where n is some integer to be specified later), and $\underline{\Omega}_{33}$ corresponds to $[\Omega_{i,j}]_n^{\infty}$. The point of introducing the operator $\hat{\Omega}$ is that $\hat{\Lambda}_0 = 0$ is the *exact* (and smallest positive) initial point of the Riccati recursion (12) for a spectral factorization of the lower right part of $\hat{\Omega}$, and leads to an outer spectral factor $\hat{W}$ such that $\hat{\Omega} = \hat{W}^* \hat{W}$, of which only the lower right part is computed. This is seen by putting $A_{-1} = 0$, $B_{-1} = 0$ in the Riccati recursion for Λ, which leads to $\hat{\Lambda}_0 = 0$. The convergence of $\hat{\Lambda}_k$ to Λ_k is studied from this observation.

As a preliminary step, the following lemma considers a special case of the above Ω.

Lemma 15. *Let be given an operator $\Omega \in \mathcal{X}$, $\Omega \gg 0$, with block decomposition*

$$\Omega = \begin{bmatrix} \underline{\Omega}_{11} & \underline{\Omega}_{12} & 0 \\ \underline{\Omega}_{12}^* & \underline{\Omega}_{22} & \underline{\Omega}_{23} \\ 0 & \underline{\Omega}_{23}^* & \underline{\Omega}_{33} \end{bmatrix}$$

where $\underline{\Omega}_{22}$ is an $n \times n$ matrix. Let the upper triangular part of Ω be locally finite and strictly stable. Then

$$(\Omega^{-1})_{33} \to (\underline{\Omega}_{33} - \underline{\Omega}_{23}^* \underline{\Omega}_{22}^{-1} \underline{\Omega}_{23})^{-1} \quad \text{as } n \to \infty$$

(strong convergence). Hence $(\Omega^{-1})_{33} \to (\hat{\Omega}^{-1})_{33}$, where $\hat{\Omega}$ is equal to Ω, but with $\hat{\underline{\Omega}}_{12} = 0$.

PROOF Let $\{A, B, C, D\}$ be a realization of the upper triangular part of Ω with $\ell_A < 1$. Let $\underline{\Omega}_{12} = \underline{\mathcal{C}}_1 \underline{\mathcal{O}}_1$, $\underline{\Omega}_{23} = \underline{\mathcal{C}}_2 \underline{\mathcal{O}}_2$, where

$$\underline{\mathcal{C}}_1 = \begin{bmatrix} \vdots \\ B_{-3}A_{-2}A_{-1} \\ B_{-2}A_{-1} \\ B_{-1} \end{bmatrix}, \qquad \underline{\mathcal{C}}_2 = \begin{bmatrix} B_0 A_1 \cdots A_{n-1} \\ \vdots \\ B_{n-3}A_{n-2}A_{n-1} \\ B_{n-2}A_{n-1} \\ B_{n-1} \end{bmatrix},$$

$$\underline{\mathcal{O}}_1 = [\, C_0 \quad A_0 C_1 \quad A_0 A_1 C_2 \quad \cdots \quad A_0 \cdots A_{n-2} C_{n-1} \,]$$

$$\underline{\mathcal{O}}_2 = [\, C_n \quad A_n C_{n+1} \quad A_n A_{n+1} C_{n+2} \quad \cdots \,].$$

Then $\underline{\mathcal{O}}_1 \underline{\mathcal{C}}_2$ is a summation of n terms, each containing a product $A_0 \cdots A_{i-1}$ and a product $A_{i+1} \cdots A_{n-1}$. Because $\ell_A < 1$ implies that products of the form $A_k \cdots A_{k+n} \to 0$ as $n \to \infty$ strongly and uniformly in k, we obtain $\underline{\mathcal{O}}_1 \underline{\mathcal{C}}_2 \to 0$ if $n \to \infty$.

Write $X_3 = (\Omega^{-1})_{33}$. By repeated use of Schur's inversion formula, X_3 is given by the recursion

$$X_1 = \underline{\Omega}_{11}^{-1}, \qquad X_{k+1} = (\underline{\Omega}_{k+1,k+1} - \underline{\Omega}_{k,k+1}^* X_k \underline{\Omega}_{k,k+1})^{-1}. \tag{21}$$

We first consider a special case, where $\underline{\Omega}_{k,k} = I$ ($k = 1, 2, 3$). In the derivation below, we, for ease of discussion, assume that also $\underline{\mathcal{O}}_k \underline{\mathcal{O}}_k^* = I$, i.e., the realization is uniformly observable and in output normal form, although this is not an essential requirement. The recursion (21) becomes

$$\begin{aligned} Y_k &= \underline{\mathcal{C}}_k^* X_k \underline{\mathcal{C}}_k \\ X_{k+1} &= (I - \underline{\mathcal{O}}_k^* Y_k \underline{\mathcal{O}}_k)^{-1} = I + \underline{\mathcal{O}}_k^* \left[Y_k + Y_k^2 + \cdots \right] \underline{\mathcal{O}}_k, \end{aligned}$$

[2] The underscore is used in this section to denote that we take block submatrices rather than entries of Ω.

so that, in particular,

$$Y_2 = \underline{\mathcal{C}}_2^* \underline{\mathcal{C}}_2 + \underline{\mathcal{C}}_2^* \underline{\mathcal{O}}_1^* \left[Y_1 (I - Y_1)^{-1} \right] \underline{\mathcal{O}}_1 \underline{\mathcal{C}}_2 \,.$$

For large n, $Y_2 \to \underline{\mathcal{C}}_2^* \underline{\mathcal{C}}_2$ and becomes independent of Y_1 and $\underline{\mathcal{C}}_1$, and

$$X_3 \to (I - \underline{\mathcal{O}}_2^* \underline{\mathcal{C}}_2^* \underline{\mathcal{C}}_2 \underline{\mathcal{O}}_2)^{-1} = (\underline{\Omega}_{33} - \underline{\Omega}_{23}^* \underline{\Omega}_{22}^{-1} \underline{\Omega}_{23})^{-1}$$

independently of $\underline{\mathcal{C}}_1$. The expression on the right-hand side is the same as the value obtained for $\underline{\mathcal{C}}_1 = 0$, i.e., $\underline{\Omega}_{12} = 0$.

The general case reduces to the above special case by a pre- and post-multiplication by

$$\begin{bmatrix} \underline{\Omega}_{11}^{-1/2} & & \\ & \underline{\Omega}_{22}^{-1/2} & \\ & & \underline{\Omega}_{33}^{-1/2} \end{bmatrix} .$$

This maps $\underline{\Omega}_{k,k}$ to I, $\underline{\mathcal{C}}_k$ to $\underline{\Omega}_{k,k}^{-1/2} \underline{\mathcal{C}}_k$, and $\underline{\mathcal{O}}_k$ to $\underline{\mathcal{O}}_k \underline{\Omega}_{k+1,k+1}^{-1/2}$. The latter two mappings lead to realizations with different B_i and C_i, but the A_i remain the same, and in particular the convergence properties of $\underline{\mathcal{C}}_2 \underline{\mathcal{O}}_1$ remain unchanged. It follows that $(\Omega^{-1})_{33} \to (\underline{\Omega}_{33} - \underline{\Omega}_{23}^* \underline{\Omega}_{22}^{-1} \underline{\Omega}_{23})^{-1}$ also in the general case.

□

We now return to the spectral factorization problem, with Ω given as in (20).

Proposition 16. *Let $\Omega \in \mathcal{X}$, $\Omega \gg 0$ have an upper triangular part which is locally finite and given by an observable realization $\{A, B, C, D\}$ where $\ell_A < 1$. Let $\Lambda \in \mathcal{D}$ be the smallest positive solution of (11) so that its entries Λ_n satisfy the recursive Riccati equation (12). Let $\hat{\Lambda}_n$ ($n \ge 0$) be the sequence obtained from the same recursion, but starting from $\hat{\Lambda}_0 = 0$.*

Then $\hat{\Lambda}_n \to \Lambda_n$ as $n \to \infty$ (strong convergence).

PROOF Let Ω, $\hat{\Omega}$ have block decompositions as in (20), where $\underline{\Omega}_{22}$ is an $n \times n$ matrix. Let $\Omega = W^* W$, $\hat{\Omega} = \hat{W}^* \hat{W}$, where $W, \hat{W}$ are outer spectral factors, then Λ, $\hat{\Lambda}$ are the controllability Gramians of the realization of W, $\hat{W}$ given in corollary 9. Denote

$$\begin{aligned} W_{12} &= \underline{\mathcal{C}}_{W,1} \underline{\mathcal{O}}_1 \\ W_{23} &= \underline{\mathcal{C}}_{W,2} \underline{\mathcal{O}}_2 \\ W_{13} &= \underline{\mathcal{C}}_{W,1} A_0 A_1 \cdots A_{n-1} \underline{\mathcal{O}}_2 \,. \end{aligned}$$

Because $\ell_A < 1$, we have that $W_{13} \to 0$ as $n \to \infty$ (strongly), so that for large enough n, $\Lambda_n \approx \underline{\mathcal{C}}_{W,2}^* \underline{\mathcal{C}}_{W,2}$ and hence

$$\begin{aligned} \underline{\Omega}_{33} &= W_{33}^* W_{33} + W_{23}^* W_{23} + W_{13}^* W_{13} \\ &\approx W_{33}^* W_{33} + \underline{\mathcal{O}}_2^* \Lambda_n \underline{\mathcal{O}}_2 \,. \end{aligned}$$

Consequently, $\underline{\mathcal{O}}_2^* (\Lambda_n - \hat{\Lambda}_n) \underline{\mathcal{O}}_2 \approx \hat{W}_{33}^* \hat{W}_{33} - W_{33}^* W_{33}$. The next step is to show that $\hat{W}_{33}^* \hat{W}_{33} - W_{33}^* W_{33} \to 0$ for large n, so that, if the realization is observable, $\hat{\Lambda}_n \to \Lambda_n$.

Let $X_3 = (W_{33}^* W_{33})^{-1}$, and $\hat{X}_3 = (\hat{W}_{33}^* \hat{W}_{33})^{-1}$. Since $\Omega^{-1} = W^{-1} W^{-*}$, and W is outer so that $W^{-1} \in \mathcal{U}$, it follows that $X_3 = (\Omega^{-1})_{33}$ and $\hat{X}_3 = (\hat{\Omega}^{-1})_{33}$. Lemma 15 proves that, if $\ell_A < 1$, then $(\Omega^{-1})_{33} \to (\hat{\Omega}^{-1})_{33}$ as $n \to \infty$, so that $X_3 \to \hat{X}_3$, and hence $\hat{\Lambda}_n \to \Lambda_n$. □

Finally, we remark that always $\hat{\Lambda}_k \leq \Lambda_k$. This is a consequence of the fact that

$$\hat{\Lambda}_k \leq \Lambda_k \quad \Rightarrow \quad \hat{\Lambda}_{k+1} \leq \Lambda_{k+1}\,, \tag{22}$$

which can be proven directly from the Riccati recursion (12) in a way similar to [5, ch. 9]. Indeed, let the matrix G_{X,Λ_k} be given by

$$G_{X,\Lambda_k} = \begin{bmatrix} X - A_k^*\Lambda_k A_k & B_k - C_k^*\Lambda_k A_k \\ B_k^* - A_k^*\Lambda_k C_k & D_k - C_k^*\Lambda_k C_k \end{bmatrix} = \begin{bmatrix} X & B_k \\ B_k^* & D_k \end{bmatrix} - \begin{bmatrix} A_k^* \\ C_k^* \end{bmatrix} \Lambda_k \left[\, A_k \quad C_k \,\right],$$

parameterized by some matrix $X = X^*$. Using Schur's complements, it follows that, if $D_k - C_k^*\Lambda_k C_k > 0$, then

$$G_{X,\Lambda_k} \geq 0 \quad \Rightarrow \quad X - A_k^*\Lambda_k A_k - \left[B_k^* - A_k^*\Lambda C_k\right](D_k - C_k^*\Lambda_k C_k)^{-1}\left[B_k - C_k^*\Lambda_k A_k\right] \geq 0\,.$$

Hence $\Lambda_{k+1} = \min\{X : G_{X,\Lambda_k} \geq 0\}$. But if $\hat{\Lambda}_k \leq \Lambda_k$, then $G_{\Lambda_{k+1},\hat{\Lambda}_k} \geq G_{\Lambda_{k+1},\Lambda_k} \geq 0$. It follows that $\Lambda_{k+1} \geq \hat{\Lambda}_{k+1}$, since $\hat{\Lambda}_{k+1}$ is the smallest matrix X for which $G_{X,\hat{\Lambda}_k} \geq 0$. This proof also supplements the remark made in section 4 that the 'smallest solution' is well defined: if Λ_k is the smallest solution at one point, the resulting diagonal operator Λ is the smallest solution at all points.

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