Joint angle and delay estimation using shift-invariance properties

Alle-Jan van der Veen, Michaela C. Vanderveen, and Arogyaswami Paulraj

Abstract— **Assuming a multipath propagation scenario, we derive a closed-form subspace-based method for the simultaneous estimation of arrival angles and path delays from measured channel impulse responses, using knowledge of the transmitted pulse shape function and assuming a uniform linear array and uniform sampling. The algorithm uses a 2-D ESPRIT-like shift-invariance technique to separate and estimate the phase shifts due to delay and direction-of-incidence, with automatic pairing of the two parameter sets. A straightforward extension to the multi-user case allows to connect rays to users as well.**

I. Introduction

One interesting problem in wireless communications is to try to estimate the angles of incidence and path delays of emitted user signals arriving at a base station antenna array, assuming that a specular multipath channel model holds true, and that the pulse shape function is known. This problem has several applications, including e.g., mobile localization for directional transmission in the down link or emergency services. It is in fact a classical radar problem.

Various approaches to the joint estimation problem with known pulse shape have been proposed in the literature (see e.g., [1–4], and references therein). These approaches often require computationally expensive ML searches and/or need accurate initial points. Assuming uniform sampling and a uniform linear array, the algorithm we develop herein transforms the data by a DFT and a deconvolution by the known pulse shape function (as in [2, 3]), and stacks the result into a Hankel matrix. This reduces the problem to one that can be solved using 2-D ESPRIT [5, 6]. Thus, the algorithm is closed-form and computationally attractive. The number of rays may be larger than the number of antennas, which overcomes a limitation of the ESPRIT method mentioned in [2] for initialization.

II. Data model

Assume we transmit a digital sequence $\{s_k\}$ over a channel, and measure the response using M antennas. The noiseless received data in general has the form $\mathbf{x}(t) = \sum_{k=1}^{N} s_k \mathbf{h}(t-kT)$, where T is the symbol rate, which will be normalized to $T = 1$ from now on. A commonly used multiray propagation model, for specular multipath, writes the $M \times 1$ channel impulse response as

$$\mathbf{h}(t) = \sum_{i=1}^{r} \mathbf{a}(\alpha_i)\beta_i g(t-\tau_i)$$

where $g(t)$ is a known pulse shape function by which $\{s_k\}$ is modulated. In this model, there are r distinct propagation paths, each parameterized by $(\alpha_i, \tau_i, \beta_i)$, where α_i is the direction-of-arrival (DOA), τ_i is the path delay, and $\beta_i \in \mathbb{C}$ is the complex path attenuation (fading). The vector-valued function $\mathbf{a}(\alpha)$ is the array response vector for an array of M antenna elements to a signal from direction α.

Suppose $\mathbf{h}(t)$ has finite duration and is zero outside an interval $[0, L)$, where L is the (integer) channel length. We assume that the received data $\mathbf{x}(t)$ is sampled at a rate P times the symbol rate. Using either training sequences (known $\{s_k\}$) or blind channel estimation techniques (e.g., [7]), it is possible to estimate $\mathbf{h}(k)$, $k = 0, \frac{1}{P}, \cdots, L-\frac{1}{P}$, at least up to a scalar.

Collect the samples of the known waveform $g(t)$ into a row vector $\mathbf{g} = [g(0) \quad g(\frac{1}{P}) \quad \cdots \quad g(L-\frac{1}{P})]$. The data model can be written as

$$H = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_r] \begin{bmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_r \end{bmatrix} \begin{bmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_r \end{bmatrix} =: ABG \qquad (1)$$

where $\mathbf{a}_i = \mathbf{a}(\alpha_i)$, and $\mathbf{g}_i = [g(t-\tau_i)]_{k=0,1/P,\cdots,L-1/P}$ is a row vector containing the samples of $g(t-\tau)$.

The delay estimation algorithm is based on the property that the Fourier transform maps a delay to a phase shift. Thus let $\tilde{\mathbf{g}} = \mathbf{g}\mathcal{F}$ where $\mathcal{F}$ denotes the DFT matrix of size $LP \times LP$, defined by

$$\mathcal{F} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \phi & \cdots & \phi^{LP-1} \\ \vdots & \vdots & & \vdots \\ 1 & \phi^{LP-1} & \cdots & \phi^{(LP-1)^2} \end{bmatrix}, \qquad \phi = e^{-j\frac{2\pi}{LP}}.$$

If τ is an integer multiple of $\frac{1}{P}$, or if $g(t)$ is bandlimited[1] and we sample at or above the Nyquist rate, then it is straightforward to see that the Fourier transform $\tilde{\mathbf{g}}_\tau$ of the sampled version of $g(t-\tau)$ is given by $\tilde{\mathbf{g}}_\tau = [1 \quad \phi^{\tau P} \quad (\phi^{\tau P})^2 \ \cdots \ (\phi^{\tau P})^{LP-1}]\,\mathrm{diag}(\tilde{\mathbf{g}})$. The same holds approximately true if τ is not an integer multiple of $\frac{1}{P}$, depending on the bandwidth of $g(t)$ and the number of

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[1] This is not in full agreement with the FIR assumption. The truncation introduces a small bias.

samples LP. Thus we can write the Fourier-transformed data model $\tilde{H} := H\mathcal{F}$ as $\tilde{H} = ABF\text{diag}(\tilde{\mathbf{g}})$, where

$$F_{LP} := \begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{LP-1} \\ \vdots & \vdots & & & \vdots \\ 1 & \phi_r & \phi_r^2 & \cdots & \phi_r^{LP-1} \end{bmatrix}, \quad \phi_i := e^{-j\frac{2\pi}{L}\tau_i}$$

(we usually omit the size index of F). Assume that $g(t)$ is bandlimited with normalized bandwidth P'. Then $\tilde{\mathbf{g}}$ has at most LP' nonzero entries, which can be selected by a selection matrix $J_{\tilde{\mathbf{g}}} : LP \times LP'$. If there are no other (intermittent) zeros, we can factor $\tilde{\mathbf{g}}$ out of $\tilde{H}$ and obtain

$$\bar{H} := H\,\mathcal{F}\,J_{\tilde{\mathbf{g}}}\,\{\text{diag}(\tilde{\mathbf{g}}J_{\tilde{\mathbf{g}}})\}^{-1}, \qquad (M \times LP').$$

which (up to a possible phase shift in B) satisfies the model

$$\bar{H} = ABF. \tag{2}$$

If $r \le M$, then it is possible to estimate the ϕ_i's and hence the τ_i's from the shift-invariance structure of F, independent of the structure of A, which is essentially the ESPRIT algorithm. To estimate the DOAs as well, we need to know the array manifold structure. For simplicity, we will assume a uniform linear array (ULA) consisting of omnidirectional elements with interelement spacing of Δ wavelengths, but other configurations are possible. The correct pairing of the τ_i's to the α_i's requires the use of ideas from 2-D DOA estimation (viz. [5,6]).

In general, the number of antennas is not large enough to satisfy $M > r$. We can avoid this problem by constructing a Hankel matrix out of $\bar{H}$.

III. Joint delay and angle estimation

A. Algorithm outline

From $\bar{H}$, construct a Hankel matrix $\mathcal{H}$ by left-shifting and stacking m copies of $\bar{H}$, yielding

$$\mathcal{H} = \begin{bmatrix} \bar{H}_{\bullet,1\cdots LP'-m+1} \\ \bar{H}_{\bullet,2\cdots LP'-m+2} \\ \vdots \\ \bar{H}_{\bullet,m\cdots LP'} \end{bmatrix}, \qquad (mM \times LP'-m+1).$$

$\mathcal{H}$ has a factorization

$$\mathcal{H} = \begin{bmatrix} A_\theta \\ A_\theta\Phi \\ A_\theta\Phi^2 \\ \vdots \\ A_\theta\Phi^{m-1} \end{bmatrix} BF = (A_\phi \diamond A_\theta)BF \tag{3}$$

where $\diamond$ denote a column-wise Kronecker product,

$$\Phi = \text{diag}[\phi_1 \cdots \phi_r], \qquad A_\phi = \begin{bmatrix} 1 & \cdots & 1 \\ \phi_1 & \cdots & \phi_r \\ \vdots & & \vdots \\ \phi_1^{m-1} & \cdots & \phi_r^{m-1} \end{bmatrix}$$

$$\Theta = \text{diag}[\theta_1 \cdots \theta_r], \qquad A_\theta = \begin{bmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_r \\ \vdots & & \vdots \\ \theta_1^{M-1} & \cdots & \theta_r^{M-1} \end{bmatrix}$$

$$\theta_i := e^{j\Delta 2\pi \sin(\alpha_i)}$$

The estimation of Φ and Θ from $\mathcal{H}$ is based on exploiting the various shift-invariant structures present in $A_\phi \diamond A_\theta$. Define selection matrices

$$J_{x\phi} := [I_{m-1} \quad 0_1] \otimes I_M, \qquad J_{x\theta} := I_m \otimes [I_{M-1} \quad 0_1],$$
$$J_{y\phi} := [0_1 \quad I_{m-1}] \otimes I_M, \qquad J_{y\theta} := I_m \otimes [0_1 \quad I_{M-1}],$$

and let $X_\phi = J_{x\phi}\mathcal{H}$, $Y_\phi = J_{y\phi}\mathcal{H}$, $X_\theta = J_{x\theta}\mathcal{H}$, $Y_\theta = J_{y\theta}\mathcal{H}$. These data matrices have the structure

$$\begin{cases} X_\phi = A'BF \\ Y_\phi = A'\Phi BF \end{cases} \qquad \begin{cases} X_\theta = A''BF \\ Y_\theta = A''\Theta BF \end{cases} \tag{4}$$

where $A' = J_{x\phi}(A_\phi \diamond A_\theta)$, $A'' = J_{x\theta}(A_\phi \diamond A_\theta)$. If dimensions are such that these are low-rank factorizations, then we can apply the 2-D ESPRIT algorithm [5,6] to estimate Φ and Θ. In particular, since

$$\begin{aligned} Y_\phi - \lambda X_\phi &= A'[\Phi - \lambda I_r]BF \\ Y_\theta - \lambda X_\theta &= A''[\Theta - \lambda I_r]BF \end{aligned}$$

the ϕ_i are given by the rank reducing numbers of the pencil (Y_ϕ, X_ϕ), whereas the θ_i are the rank reducing numbers of (Y_θ, X_θ). These are the same as the nonzero eigenvalues of $X_\phi^\dagger Y_\phi$ and $X_\theta^\dagger Y_\theta$. ($\dagger$ denotes the Moore-Penrose pseudo-inverse.)

The correct pairing of the ϕ_i with the θ_i follows from the fact that $X_\phi^\dagger Y_\phi$ and $X_\theta^\dagger Y_\theta$ have the same eigenvectors, which is caused by the common factor F. In particular, there is an invertible matrix V which diagonalizes both $X_\phi^\dagger Y_\phi$ and $X_\theta^\dagger Y_\theta$. Various algorithms have been derived to compute such joint diagonalizations. Omitting further details, we propose to use the diagonalization method in [5], although the algorithm in [6] can be used as well. As in ESPRIT, the actual algorithm has an intermediate step in which $\mathcal{H}$ is reduced to its r-dimensional principal column span, and this step will form the main computational bottleneck.

Once the DOAs and delays are known, the fading coefficients can be estimated straightforwardly: rewrite (1) as $\text{vec}(H) = (G^T \diamond A)\boldsymbol{\beta}$ so that

$$\boldsymbol{\beta} = (G^T \diamond A)^\dagger \text{vec}(H).$$

The fading coefficients can be used to separate multiple users, as demonstrated in section IV

B. Data extension

Since the eigenvalues (ϕ_i, θ_i) are on the unit circle, we can double the dimension of $\mathcal{H}$ by forward-backward averaging. In particular, let J denote the exchange matrix which reverses the ordering of rows, and define

$$\mathcal{H}_e = [\mathcal{H} \quad J\mathcal{H}^{(c)}] , \qquad (mM \times 2(LP' - m + 1)) , \quad (5)$$

where (c) indicates taking the complex conjugate. Since $J(A_\phi \diamond A_\theta)^{(c)} = (A_\phi \diamond A_\theta)\Phi^{-m}\Theta^{-M}$, it follows that $\mathcal{H}_e$ has a factorization

$$\mathcal{H}_e = (A_\phi \diamond A_\theta) B_e F_e = (A_\phi \diamond A_\theta)[BF , \quad \Phi^{-m}\Theta^{-M}B^{(c)}F^{(c)}] .$$

The computation of Φ and Θ from $\mathcal{H}_e$ proceeds as before. It is at this point possible to do a simple transformation to map $\mathcal{H}_e$ to a real matrix, which will keep all subsequent matrix operations real as well. This has numerical and computational advantages and is detailed in [6].

C. Identifiability

To identify Φ and Θ from (4), F should be "wide", and A' and A'' should be "tall", i.e., $r \le 2(LP' - m + 1)$, $r \le (M-1)m$, $r \le (m-1)M$. Elimination of m produces the necessary condition

$$\begin{cases} r \le 2LP' \frac{M}{M+2} , & \text{if } LP' \le \frac{1}{2}(M-1)(M+2) , \\ r \le 2(LP'+1)\frac{M-1}{M+1} , & \text{otherwise} \end{cases}$$

which gives an upper bound on the number of rays that can be estimated using this technique for given LP' and M. Equal delays or angles are acceptable, but for identifiability, it is necessary that the total number of rays with (almost) equal delays is less than M, and that the total number of rays with (almost) equal angles is less than m, otherwise A' or A'' will be singular (or badly conditioned).

D. Cramer-Rao bound

The Cramer-Rao bound (CRB) provides a lower bound on the variance of any unbiased estimator. The bound for DOA estimation (without delay spread) was derived in [8], and is readily adapted to the present situation. Assuming the path fadings to be deterministic but unknown, we obtain for the model in (1) that

$$\mathbf{CRB}(\boldsymbol{\alpha}, \boldsymbol{\tau}) = \frac{\sigma_h^2}{2} \left\{ \text{real}\left(\mathcal{B}^* D^* P_U^\perp D \mathcal{B} \right) \right\}^{-1} \qquad (6)$$

where σ_h^2 is the variance of the noise on the entries of $\mathbf{h}(t)$ (assumed to be i.i.d. white Gaussian noise), $\mathcal{B} = I_2 \otimes B$, $U = A(\boldsymbol{\alpha}) \diamond G^T(\boldsymbol{\tau})$, $P_U^\perp = U(U^*U)^{-1}U^*$, and $D = [A'(\boldsymbol{\alpha}) \diamond G^T(\boldsymbol{\tau}), A(\boldsymbol{\alpha}) \diamond G'(\boldsymbol{\tau})^T]$ (prime denotes differentiation, where each column is differentiated with respect to the corresponding parameter and all matrices are evaluated at the true parameter values).

IV. Blind multi-user separation

In blind signal separation, we do not have the channel impulse response $\mathbf{h}(t)$ available, nor the input data $\{s_k\}$. A number of techniques have been developed to estimate both parts from the observed data. One among several techniques to estimate $\mathbf{h}(t)$, up to a scaling, appears in [9]. An extension of their algorithm to the multi-user case (d users, say) is straightforward (viz. [7]), but the ambiguity becomes a constant invertible $d \times d$ matrix T: we can only estimate an arbitrary *basis* of the space spanned by the d channel impulse responses. Placed in the notation of this paper, this means that we can construct a matrix V : $dM \times LP$, with assumed model (generalized from (1))

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_d \end{bmatrix} = (T \otimes I_M) \begin{bmatrix} H_1 \\ \vdots \\ H_d \end{bmatrix} = (T \otimes I_M) \begin{bmatrix} A_1 B_1 G_1 \\ \vdots \\ A_d B_d G_d \end{bmatrix} . \qquad (7)$$

To solve the ambiguity and hence to separate the users, all blind algorithms so far have relied on properties of the signal matrix, such as its finite alphabet, or constant modulus properties. It is interesting to note that separation can also be achieved directly from the above model. The assumption is that each estimated ray belongs to only one user.

Thus suppose that we have obtained the basis V. From (7), each V_k has model

$$V_k = [A_1 \ \cdots \ A_d] \begin{bmatrix} t_{k1}B_1 & & 0 \\ & \ddots & \\ 0 & & t_{kd}B_d \end{bmatrix} \begin{bmatrix} G_1 \\ \vdots \\ G_d \end{bmatrix} =: \tilde{A}\tilde{B}_k\tilde{G} \qquad (8)$$

and hence contains a mixture of all rays, with fadings $\tilde{B}_k$. Note that we can estimate (α_i, τ_i) much as before. Indeed, after the DFT and deconvolution, we construct Hankel matrices $\mathcal{V}_1, \cdots, \mathcal{V}_d$, with assumed models $\mathcal{V}_k = (A_\phi \diamond A_\theta)\tilde{B}_k\tilde{F}$, and collect all data in $\mathcal{V}_e = [\mathcal{V}_{1,e} \cdots \mathcal{V}_{d,e}]$, which replaces $\mathcal{H}_e$ in the 2-D ESPRIT algorithm. The joint estimation of the ray parameters (including fadings) produces $\tilde{A}\Pi$, $\Pi^*\tilde{B}_k\Pi$, $\Pi\tilde{G}$, where a permutation Π accounts for the fact that we don't know the assignment of rays to users.

The diagonal entries of $\Pi^*\tilde{B}_k\Pi$, put in rows and stacked in a matrix, satisfy (viz. (8))

$$\begin{bmatrix} \text{diag}(\tilde{B}_1) \\ \vdots \\ \text{diag}(\tilde{B}_d) \end{bmatrix} \Pi = T \begin{bmatrix} \text{diag}(B_1) & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & \text{diag}(B_d) \end{bmatrix} .$$

Since each column in the matrix at the right has precisely one nonzero entry, the columns in the matrix at the left can have only d distinct directions, which are the directions of the columns of T. It suffices to determine these directions, after which Π and the membership of each ray is known. This will also determine T up to a scaling of its columns (which is the best we can hope for). Thus, the ray assignment simply calls for a normalization of the length of each column, followed by a clustering into d distinct directions.

V. Simulation results

To illustrate the performance of the algorithm, we report some computer simulation results. Here, we assume

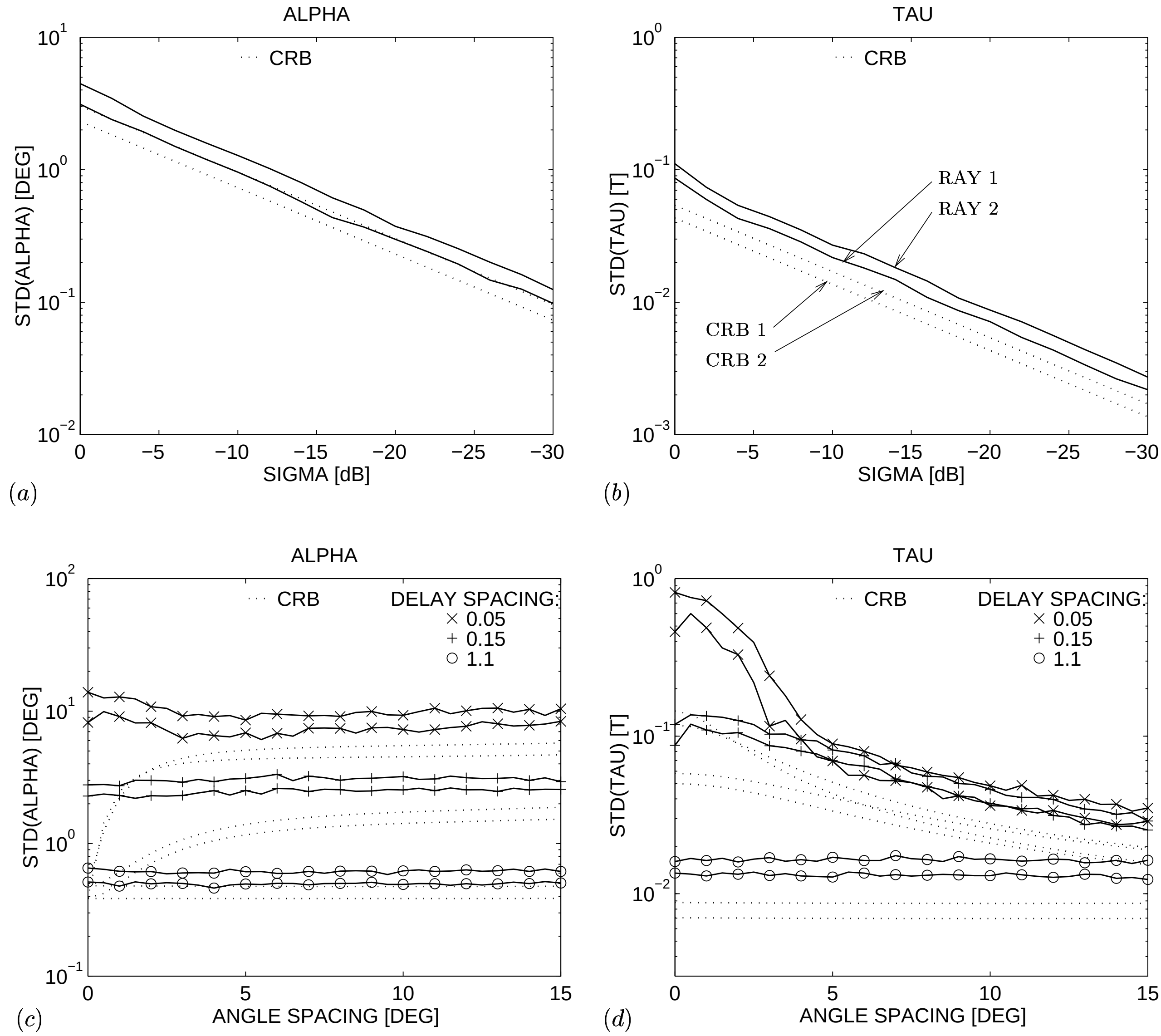

Fig. 1. Standard deviation of estimates: (a, b) varying noise power, (c, d) varying angle and delay separation, with $\sigma_x = -15$ dB.

one user and an array of $M = 2$ sensors. We also assume the communication protocol uses $N = 40$ training bits, from which the channel is estimated using least squares. The pulse shape function is a raised cosine with 0.35 excess bandwidth, truncated to a length of $L_0 = 6$ symbols. Figure 1 shows the experimental variance of the DOA and delay estimates as a function of standard deviation σ_x of the (i.i.d. white Gaussian) noise on the received data, for a scenario with $r = 2$ paths with angles $[-10, 20]^\circ$, delays $[0, 1.1]T$, fading amplitudes $[1, 0.8]$, a randomly selected but constant fading phase, stacking parameter $m = 3$, and $P = 2$ times oversampling. It is seen that the difference in performance compared to the CRB is approximately 4 dB. The bias of the estimates was at least an order of magnitude smaller than their standard deviation.

The achievable resolution is demonstrated by varying the DOA and delay of the second ray, keeping the DOA and delay of the first ray fixed at $(-10^\circ, 0T)$. The same parameters as before were used, with noise power -15 dB. As expected, the performance in comparison to the CRB suffers when both τ's and α's are closely spaced, since with two antennas we cannot separate two rays with identical delays using ESPRIT.

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