COMPUTATION OF THE INNER-OUTER FACTORIZATION FOR TIME-VARYING SYSTEMS

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Abstract

An inner-outer factorization theorem for linear time-varying systems is obtained via an extension of the classical Beurling-Lax theorem to the time-varying context. This provides characteristic features of the inner factor, which can be used to compute realizations of the inner and outer factors from a realization of the given transfer operator. The resulting algorithm is in essence the square-root version of a Riccati recursive equation with time-varying coefficients.

1. Introduction

Recently, there has been some interest in computing inner-outer factorizations of time-varying systems, as a basic step in robust control applications such as the design of feedback controllers and sensitivity minimization [6, 2]. For *time-invariant* single-input single-output systems, the inner-outer factorization is a factorization of an analytical (causal) transfer function $T(z)$ into the product of an inner and an outer system: $T(z) = V(z)T_0(z)$. The inner part $V(z)$ is analytical (*i.e.,* it has its poles outside the unit disc) and has modulus 1 on the unit circle, whereas the outer part $T_0(z)$ is analytical and may have zeros only outside the unit disc. For example, (with $|\alpha|, |\beta| < 1$)

$$z\,\frac{z-\overline{\alpha}}{1-\beta z} \;=\; z\,\frac{z-\overline{\alpha}}{1-\alpha z}\cdot\frac{1-\alpha z}{1-\beta z}\,.$$

The location of the zeros determines the inner and outer factors. The resulting outer factor is such that its inverse is again a stable system, provided there are no zeros on the unit circle. For multi-input multi-output systems, the definition of the outer factor is more abstract (see *e.g.,* Halmos [7]) and takes the form of a range condition: $T_0(z)$ is outer if $T_0(z)H_m^2 = H_n^2$, where H_m^2 is the Hardy space

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of analytical m-dimensional vector-valued functions. A generalization of this definition applies in the time-varying context.

The existence of inner-outer factorizations in any context is more or less fundamental to analytical Hilbert spaces. Abstract mathematical formulations of it which also apply to the time-varying setting can be found in [1, 8]. In this paper, we connect the abstract theory to a computational scheme acting on state space realizations. One of the aspects of time-varying systems is that the state dimension can vary, and as a result, the number of 'zeros' in the inner and outer factors can vary, too. The theory in this paper handles such variations automatically. Full details can be found in [10].

An application of the inner-outer factorization is the computation of inverse systems: if T is a causal and invertible system, then its inverse is not necessarily causal: the inversion might have introduced an anti-causal part. This effect is known as a dichotomy; it is in general not a trivial task to determine the causal and anti-causal parts of T^{-1}. With the inner-outer factorization, however, the inverse of the outer factor is again causal, whereas the inverse of the inner factor is fully anti-causal, and determines which part of the inverse of the outer factor is made anti-causal. This application of the inner-outer factorization plays a crucial role in *e.g.*, the computation of optimal feedback controllers [6].

Of less importance to control, but for the sake of illustration, applications to finite matrix computations are considered. The inner-outer factorization can be viewed as a state space technique to efficiently compute a QR factorization of a structured matrix.

2. Notation

We adopt the notation of [3, 4, 10], so that the description of it in this paper will be terse. For $i = \cdots, -1, 0, 1, \cdots$, let $\mathcal{M}_i$ be a separable Hilbert space. We will usually take $\mathcal{M}_i = \mathbb{C}^{M_i}$ with M_i a finite number, so that $\mathcal{M}_i$ is a finite vector space, but we will have to allow $\mathcal{M}_i = \ell_2$ in section 5. The space $\mathcal{M} = \mathbb{C}^M := \cdots \times \mathcal{M}_0 \times \mathcal{M}_1 \times \cdots$ is the space of (non-uniform) sequences $u = [\cdots, u_0, u_1, \cdots]$ with entries u_i in $\mathcal{M}_i$. Such sequences will represent the signals in our systems. If all dimensions M_i are finite, then we call $\mathcal{M}$ *locally finite*. Some (or most) of the dimensions may be zero, and in this way finite non-uniform vectors are also included in the formalism. In $\mathcal{M}$, the inner product between two non-uniform sequences f and g is defined in terms of the usual inner product of (row)-vectors in $\mathcal{M}_i$ as

$$(f, g) = \sum_i (f_i, g_i)$$

where $(f_i, g_i) = f_i g_i^*$ is defined to be 0 if $M_i = 0$. The norm of a non-uniform sequence is the standard 2-norm (vector norm) defined on this inner product:

$$u = [\, u_i \,]_{-\infty}^{\infty} \;:\quad \|\, u \,\|_2^2 = (u, u) = \sum_{i=-\infty}^{\infty} \|\, u_i \,\|_2^2 \,.$$

The space of sequences in $\mathcal{M}$ with finite 2-norm is denoted by $\ell_2^{\mathcal{M}}$:

$$\ell_2^{\mathcal{M}} = \{u \in \mathcal{M} : \|\, u \,\|_2 < \infty\} \,.$$

The space $\mathcal{X}(\mathcal{M}, \mathcal{N})$ is the space of bounded operators $T : y = uT$ acting from $\ell_2^{\mathcal{M}}$ into $\ell_2^{\mathcal{N}}$.[1] An operator in such a space has an (infinite) matrix representation where the ij-th entry is an operator $\mathcal{M}_i \to \mathcal{N}_j$ (an $M_i \times N_j$ matrix). The space $\mathcal{U} \subset \mathcal{X}$ consists of bounded operators T which are upper:

[1] For historical reasons, the ordering of operators is written from left to right throughout the paper.

$T_{ij} = 0$ $(i > j)$. Likewise, we define $\mathcal{L} \subset \mathcal{X}$ to be the space of lower operators and $\mathcal{D} = \mathcal{U} \cap \mathcal{L}$ to be the space of diagonals.

In $\mathcal{X}$, the causal bilateral shift-operator Z is defined via $[\cdots, \boxed{u_0}, u_1, \cdots]Z = [\cdots, \boxed{u_{-1}}, u_0, \cdots]$ (the square identifies the position of the 0-th entry). $Z \in \mathcal{U}$, and Z has a matrix representation in which the entries $Z_{i,i+1} = I$, and all other entries are zero. If $u \in \mathcal{M}$, then $uZ \in \mathcal{M}^{(1)}$, where $\mathcal{M}^{(1)}$ is equal to the space sequence $\mathcal{M}$, shifted over one position. The k-th diagonal shift of an operator A into the South-East direction is denoted as $A^{(k)} = Z^{*k}AZ^k$.

Based on the space $\mathcal{X}(\mathcal{M}, \mathcal{N})$ of $[\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}]$ operators bounded in operator norm, we define the Hilbert-Schmidt space $\mathcal{X}_2(\mathcal{M}, \mathcal{N})$, consisting of elements of $\mathcal{X}$ that are also bounded in Hilbert-Schmidt norm. This norm is defined as

$$\| A \|_{HS}^2 = \sum_{i,j} \|A_{ij}\|_2^2 \qquad (A \in \mathcal{X}(\mathcal{M}, \mathcal{N})),$$

where $\|A_{ij}\|_2^2$ is, in turn, equal to the sum of the entries of A_{ij} squared. $\mathcal{X}_2(\mathcal{M}, \mathcal{N})$ is thus given by

$$\mathcal{X}_2(\mathcal{M}, \mathcal{N}) \;=\; \{A \in \mathcal{X}(\mathcal{M}, \mathcal{N}) : \; \| A \|_{HS}^2 < \infty\}.$$

On $\mathcal{X}_2(\mathcal{M}, \mathcal{N})$, the Hilbert-Schmidt inner product

$$\langle A, B \rangle_{HS} = \operatorname{trace}(AB^*)$$

can be defined, and the Hilbert Schmidt norm satisfies $\| A \|_{HS}^2 = \langle A, A\rangle_{HS} = \operatorname{trace}(AA^*)$. $\mathcal{X}_2(\mathcal{M}, \mathcal{N})$ is a Hilbert space for the Hilbert-Schmidt inner product. We will only use the special case $\mathcal{X}_2^{\mathcal{M}} := \mathcal{X}_2(\mathbb{C}^{\mathbb{Z}}, \mathcal{M})$. It can be considered as the input or output space for our system operators: each row of an element of $\mathcal{X}_2^{\mathcal{M}}$ is a sequence in $\ell_2^{\mathcal{M}}$, and if T is a bounded operator $[\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}]$, then it may be extended as a bounded operator $[\mathcal{X}_2 \to \mathcal{X}_2]$ by stacking sequences in ℓ_2 to form elements of $\mathcal{X}_2$. This leads for example to the expression $Y = UT$, where $U \in \mathcal{X}_2^{\mathcal{M}}$ and $Y \in \mathcal{X}_2^{\mathcal{N}}$ [3]. If the k-th row of U is denoted by U_k (and likewise for Y), then $Y = UT$ corresponds to $Y_k = U_k T$, for $k = \cdots, 0, 1, \cdots$. We will continue to write $\mathcal{X}_2$ instead of $\mathcal{X}_2^{\mathcal{M}}$ if the precise form of $\mathcal{M}$ is not of interest.

We define $\mathbf{P}$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{U}_2$, $\mathbf{P}_0$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{D}_2$, and $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{L}_2 Z^{-1}$. If $X \in \mathcal{X}_2$, then its k-th diagonal is defined in terms of $\mathbf{P}_0$ by $X_{[k]} = \mathbf{P}_0(Z^{-k}X)$, and $X = \sum Z^k X_{[k]}$. The domain of the projector $\mathbf{P}_0$ can be extended to operators in $\mathcal{X}$.

A subspace $\mathcal{H}$ of $\mathcal{X}_2^{\mathcal{M}}$ is called (left) D-invariant if $D\mathcal{H} \subset \mathcal{H}$ for all diagonals $D \in \mathcal{D}$, and shift-invariant if $Z\mathcal{H} \subset \mathcal{H}$. For a D-invariant subspace $\mathcal{H}$, if $X \in \mathcal{H}$, then the rows X_k of X are elements of a subspace $\mathcal{H}_k$ in $\ell_2^{\mathcal{M}}$. Hence, a D-invariant subspace $\mathcal{H}$ falls apart into rows $\mathcal{H}_k$ such that $\mathcal{H} = \cdots \times \mathcal{H}_0 \times \mathcal{H}_1 \times \cdots$, where each $\mathcal{H}_k$ is a subspace in $\ell_2^{\mathcal{M}}$ (the k-th row of $\mathcal{X}_2^{\mathcal{M}}$) [11, 4, 10]. Let d_k be the dimension of $\mathcal{H}_k$, then we call $d = [d_k]_{-\infty}^{\infty}$ the sequence of dimensions of $\mathcal{H}$: $d = \text{s-dim}\,\mathcal{H}$. If all d_k are finite, then $\mathcal{H}$ is said to be *locally finite*. Let $\mathcal{B}_k = \mathbb{C}^{d_k}$ ($\mathcal{B}_k = \ell_2$ if $d_k = \infty$), and let $\mathcal{B} = \cdots \times \mathcal{B}_k \times \cdots$. Each $\mathcal{H}_k$ has a basis representation $\mathbf{F}_k$ such that $\mathcal{H}_k = \mathcal{B}_k \mathbf{F}_k$, where the rows of $\mathbf{F}_k$ are the individual basis vectors. Likewise, a D-invariant subspace $\mathcal{H}$ has a basis representation $\mathbf{F}$ such that $\mathcal{H} = \mathcal{D}_2 \mathbf{F}$ [11], where the k-th (block)-row of $\mathbf{F}$ is $\mathbf{F}_k$. The diagonal operator $\Lambda_{\mathbf{F}} := \mathbf{P}_0(\mathbf{F}\mathbf{F}^*) = \operatorname{diag}[\mathbf{F}_k \mathbf{F}_k^*]_{-\infty}^{\infty}$ plays the role of Gram operator. If $\Lambda_{\mathbf{F}}$ is uniformly positive (*i.e.*, boundedly invertible), the basis representation is called *strong* (it is a Riesz basis), and the projection operator onto $\mathcal{H}$ is in this case given by $\mathbf{P}_{\mathcal{H}}(\,\cdot\,) = \mathbf{P}_0(\,\cdot\,\mathbf{F}^*)\Lambda_{\mathbf{F}}^{-1}\mathbf{F}$. If $\Lambda_{\mathbf{F}} = I$, then $\mathbf{F}$ is called an orthonormal basis representation.

3. Time-varying systems

An upper operator $T \in \mathcal{U}(\mathcal{M},\mathcal{N})$ is called a causal transfer operator: it maps sequences $u \in \ell_2^{\mathcal{M}}$ to sequences $y = uT \in \ell_2^{\mathcal{N}}$ in a causal way. When T is viewed as an operator from $\mathcal{X}_2^{\mathcal{M}}$ to $\mathcal{X}_2^{\mathcal{N}}$, then because $\mathcal{X}_2 = \mathcal{L}_2 Z^{-1} \oplus \mathcal{U}_2$, its action on $\mathcal{L}_2 Z^{-1}$ can be decomposed into two operators H_T and K_T:

$$\cdot\, T|_{\mathcal{L}_2 Z^{-1}} = \cdot K_T + \cdot H_T : \qquad \cdot H_T = \mathbf{P}(\cdot\, T|_{\mathcal{L}_2 Z^{-1}})\,; \quad \cdot K_T = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\cdot\, T|_{\mathcal{L}_2 Z^{-1}}) \tag{1}$$

H_T is called the Hankel operator of T. The space $\mathcal{L}_2 Z^{-1}$ can be viewed as a 'past signal space': it consists of a collection (stack) of ℓ_2-sequences such that the k-th sequence (row) is zero from point k on. Likewise, the space $\mathcal{U}_2$ is thought of as a 'future signal space', because its k-th row of an operator in $\mathcal{U}_2$ is zero before point k. Considering such spaces leads to a mathematically compact (index-free) notation.

The range and kernel of H_T and H_T^* are D-invariant subspaces with important system-theoretic properties [11]: define

$$\begin{array}{lcccl} \mathcal{K}(T) & := & \ker(\cdot H_T) & = & \{U \in \mathcal{L}_2 Z^{-1} : \mathbf{P}(UT) = 0\} \\ \mathcal{H}(T) & := & \operatorname{ran}(\cdot H_T^*) & = & \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 T^*) \\ \mathcal{H}_0(T) & := & \operatorname{ran}(\cdot H_T) & = & \mathbf{P}(\mathcal{L}_2 Z^{-1} T) \\ \mathcal{K}_0(T) & := & \ker(\cdot H_T^*) & = & \{Y \in \mathcal{U}_2 : \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(YT^*) = 0\}\,. \end{array}$$

These subspaces provide decompositions of $\mathcal{L}_2 Z^{-1}$ and $\mathcal{U}_2$ as

$$\begin{array}{ccccl} \overline{\mathcal{H}} & \oplus & \mathcal{K} & = & \mathcal{L}_2 Z^{-1} \\ \overline{\mathcal{H}}_0 & \oplus & \mathcal{K}_0 & = & \mathcal{U}_2\,, \end{array}$$

(the overbar denotes closure). $\mathcal{H}(T)$ is called the (natural) input state space, and $\mathcal{H}_0(T)$ the (natural) output state space of T. If these subspaces are locally finite, then they have the same s-dimension, and T is said to be *locally finite*. In this case, one can obtain minimal realizations of the type

$$uT = y \qquad \Leftrightarrow \qquad \begin{array}{lcl} xZ^{-1} & = & xA + uB \\ y & = & xC + uD \end{array} \qquad \mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \tag{2}$$

where $u \in \ell_2^{\mathcal{M}}$, $y \in \ell_2^{\mathcal{N}}$,

$$\begin{array}{lcll} A & \in & \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)})\,, & C \in \mathcal{D}(\mathcal{B},\mathcal{N})\,, \\ B & \in & \mathcal{D}(\mathcal{M},\mathcal{B}^{(-1)})\,, & D \in \mathcal{D}(\mathcal{M},\mathcal{N})\,. \end{array} \tag{3}$$

The space sequence $\mathcal{B}$ is called the system order of the realization. Let A_k be the k-th entry of the diagonal operator A, and likewise for B_k, C_k, D_k. If $\mathcal{B}_k = \mathbb{C}^{d_k}$, then A_k is a $d_k \times d_{k+1}$ matrix, $B_k : M_k \times d_{k+1}$, $C_k : d_k \times N_k$, $D_k : M_k \times N_k$, and all dimensions are time-varying. By taking the k-th entry of each sequence and diagonal in (2), it is seen that these realization equations are equivalent to the (more familiar) formulation

$$\begin{array}{lcl} x_{k+1} & = & x_k A_k + u_k B_k \\ y_k & = & x_k C_k + u_k D_k \end{array} \qquad k = \cdots, 0, 1, \cdots.$$

Alternatively, we may consider the input-output relations $Y = UT$, where $U \in \mathcal{X}_2^{\mathcal{M}}$ and $Y \in \mathcal{X}_2^{\mathcal{N}}$ are collections of signals in ℓ_2, which leads to state equations in which all variables are diagonal operators:

$$\begin{array}{lcl} X_{[i+1]}^{(-1)} & = & X_{[i]}A + U_{[i]}B \\ Y_{[i]} & = & X_{[i]}C + U_{[i]}D \end{array} \qquad i = \cdots, 0, 1, \cdots. \tag{4}$$

$\mathbf{T}$ is a realization of T if its entries T_{ij} or its diagonals $T_{[i]}$ are given by

$$T_{ij} = \begin{cases} 0\,, & i > j \\ D_i\,, & i = j \\ B_i A_{i+1} \cdots A_{j-1} C_j\,, & i < j\,. \end{cases} \qquad \Leftrightarrow \qquad T_{[i]} = \begin{cases} 0, & i < 0 \\ D, & i = 0 \\ B^{(i)} A^{(i-1)} \cdots A^{(1)} C, & i > 0 \end{cases} \tag{5}$$

Let ℓ_A denote the spectral radius of the operator AZ:

$$\ell_A = \lim_{n\to\infty} \| (AZ)^n \|^{1/n}\,.$$

A realization is said to be strictly stable if $\ell_A < 1$. In this case, $(I - AZ)$ is invertible, and (2) yields $T = D + BZ(I - AZ)^{-1}C$. For a strictly stable system, the operators $\mathbf{F}$ and $\mathbf{F}_0$ defined by

$$\begin{array}{lcl} \mathbf{F} & := & \left(BZ + BZAZ + BZ(AZ)^2 + \cdots \right)^* \\ \mathbf{F}_0 & := & C + AZC + (AZ)^2 C + \cdots \end{array} \tag{6}$$

are bounded operators in $\mathcal{L}Z^{-1}$ and $\mathcal{U}$, respectively, and given by $\mathbf{F} = \left(BZ(I - AZ)^{-1}\right)$ and $\mathbf{F}_0 = (I - AZ)^{-1}C$, respectively. In case $\ell_A \le 1$, then $\mathbf{F}$ and $\mathbf{F}_0$ are bounded operators on $\mathcal{D}_2$, and can be defined via (6) on a dense subset of $\mathcal{X}_2$.

$\mathbf{F}$ and $\mathbf{F}_0$ play the role of controllability and observability operators, as they generate a factorization of the Hankel operator H_T defined in (1) as

$$H_T :\ \mathcal{L}_2 Z^{-1} \to \mathcal{U}_2, \qquad H_T = \mathbf{P}_0(\,\cdot\,\mathbf{F}^*)\,\mathbf{F}_0\,.$$

The realization is called (uniformly) controllable if the controllability Gramian $\Lambda_{\mathbf{F}} := \mathbf{P}_0(\mathbf{F}\mathbf{F}^*)$ is (uniformly) positive, (uniformly) observable if the observability Gramian $\Lambda_{\mathbf{F}_0} = \mathbf{P}_0(\mathbf{F}_0\mathbf{F}_0^*)$ is (uniformly) positive, and minimal if it is both controllable and observable. Equivalently, a realization is controllable if $\cdot\,\mathbf{F}|_{\mathcal{D}_2}$ is one-to-one (injective), *i.e.*, if $\mathbf{P}_0(\,\cdot\,\mathbf{F}^*)$ has an empty kernel, and observable if $\cdot\,\mathbf{F}_0|_{\mathcal{D}_2}$ is one-to-one. For minimal realizations, $\mathbf{F}$ and $\mathbf{F}_0$ are basis representations of $\overline{\mathcal{H}}(T)$ and $\overline{\mathcal{H}}_0(T)$, respectively:

$$\overline{\mathcal{H}}(T) = \overline{\mathcal{D}_2^{\mathcal{B}}\mathbf{F}}\,, \qquad \overline{\mathcal{H}}_0(T) = \overline{\mathcal{D}_2^{\mathcal{B}}\mathbf{F}_0}\,.$$

More in general, for a controllable realization, $\mathcal{H}_0(T) \subset \mathcal{D}_2\mathbf{F}_0$, and for an observable realization, $\mathcal{H}(T) \subset \mathcal{D}_2\mathbf{F}$. We mention the following properties, which are valid for $\ell_A \le 1$ [11, 10]:

$$\left\{\begin{array}{lcl} \mathbf{F}_0 & = & C + AZ\mathbf{F}_0 \\ T & = & D + BZ\mathbf{F}_0 \end{array}\right. \qquad \left\{\begin{array}{lcl} Z\mathbf{F} & = & A^*\mathbf{F} + B^* \\ T^* & = & D^* + C^*\mathbf{F} \end{array}\right. \qquad \left\{\begin{array}{lcl} \Lambda_{\mathbf{F}_0} = I & \Rightarrow & AA^* + CC^* = I \\ \Lambda_{\mathbf{F}} = I & \Rightarrow & A^*A + B^*B = I \end{array}\right. \tag{7}$$

4. State space properties of inner systems

A transfer operator V of a system is an isometry if $VV^* = I$, a co-isometry if $V^*V = I$, and unitary if both $VV^* = I$ and $V^*V = I$. A system is inner if its transfer operator is unitary and upper. A realization $\mathbf{V}$ is called unitary (or lossless) if $\mathbf{V}\mathbf{V}^* = I$ and $\mathbf{V}^*\mathbf{V} = I$, where

$$\mathbf{V} = \begin{bmatrix} A_V & C_V \\ B_V & D_V \end{bmatrix}\,. \tag{8}$$

Proposition 1. *Let $V \in \mathcal{U}$. Then*

$$VV^* = I \quad \Rightarrow \quad \mathcal{K}_0(V) = \mathcal{U}_2 V \oplus \ker(\,\cdot\, V^*|_{\mathcal{U}_2})\,.$$

If $VV^* = I$ and $\ker(\,\cdot\, V^*|_{\mathcal{U}_2}) = 0$ then V is inner. Dually,

$$V^*V = I \quad \Rightarrow \quad \mathcal{K}(V) = \mathcal{L}_2 Z^{-1} V^* \oplus \ker(\,\cdot\, V|_{\mathcal{L}_2 Z^{-1}})\,.$$

If $V^*V = I$ and $\ker(\,\cdot\, V|_{\mathcal{L}_2 Z^{-1}}) = 0$, then V is inner.

Proof Let $VV^* = I$. Because V is an isometry, the subspaces $\mathcal{X}_2 V = \mathrm{ran}(V)$, $\mathcal{L}_2 Z^{-1} V$ and $\mathcal{U}_2 V$ are closed, and $\mathcal{X}_2 V = \mathcal{L}_2 Z^{-1} V \oplus \mathcal{U}_2 V$. $\mathcal{U}_2 V \subset \mathcal{K}_0$, because $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,[\mathcal{U}_2 V]\, V^*) = 0$. The remaining subspace $\mathcal{K}_0 \ominus \mathcal{U}_2 V$ consists of elements

$$\begin{aligned}
\mathcal{K}_0 \ominus \mathcal{U}_2 V &= \{X \in \mathcal{U}_2 : \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(XV^*) = 0 \ \wedge\ \mathbf{P}(XV^*) = 0\} \\
&= \{X \in \mathcal{U}_2 : XV^* = 0\} \\
&= \ker(\,\cdot\, V^*|_{\mathcal{U}_2})\,.
\end{aligned}$$

Hence $\mathcal{K}_0 = \mathcal{U}_2 V \oplus \ker(\,\cdot\, V^*|_{\mathcal{U}_2})$.

If $\ker(\,\cdot\, V^*|_{\mathcal{U}_2}) = 0$, then $X \in \mathcal{U}_2\,,\ XV^* = 0 \ \Rightarrow\ X = 0$. This implies

$$X \in Z^{-n}\mathcal{U}_2\,,\ XV^* = 0 \quad \Rightarrow \quad X = 0 \qquad (\text{all } n \in \mathbb{Z})\,,$$

since $(Z^n X)V^* = 0 \ \Leftrightarrow\ XV^* = 0$. Letting $n \to \infty$ yields $\ker(\,\cdot\, V^*) = 0$, so that V has a left inverse, which must be equal to the right inverse V^*. Hence $V^*V = I$ and V is inner. Dual results hold in case $V^*V = I$. □

Unitary realizations and inner systems are closely connected: one can show that a locally finite inner system has a unitary realization, and the converse is true at least when the realization is strictly stable. More precisely, we have the following theorem.

Theorem 2. *Let $\mathbf{V}$ given by equation (8) be a state realization of a bounded transfer operator $V \in \mathcal{U}(\mathcal{M}, \mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are locally finite spaces of sequences. Let $\Lambda_{\mathbf{F}}$ and $\Lambda_{\mathbf{F}_0}$ be the controllability and the observability Gramians of the given realization. If $\ell_{A_V} < 1$, then*

$$\begin{aligned}
\mathbf{V}^*\mathbf{V} = I \quad &\Rightarrow \quad V^*V = I\,, \quad \Lambda_{\mathbf{F}} = I\,, \\
\mathbf{V}\mathbf{V}^* = I \quad &\Rightarrow \quad VV^* = I\,, \quad \Lambda_{\mathbf{F}_0} = I\,.
\end{aligned} \tag{9}$$

If $\ell_{A_V} \le 1$, then

$$\begin{aligned}
\mathbf{V}^*\mathbf{V} = I\,, \quad \Lambda_{\mathbf{F}} = I \quad &\Rightarrow \quad V^*V = I\,, \\
\mathbf{V}\mathbf{V}^* = I\,, \quad \Lambda_{\mathbf{F}_0} = I \quad &\Rightarrow \quad VV^* = I\,.
\end{aligned}$$

Proof See [4, 10]. □

5. Beurling-Lax type theorem

Proposition 1 states that the output null space of an isometric system has a certain structure. The following theorem provides a converse of this. In its time-invariant form, such a theorem is known as the Beurling-Lax theorem.

Theorem 3. *All DZ-invariant subspaces $\mathcal{K}_0$ in $\mathcal{U}_2^{\mathcal{N}}$ have the form $\mathcal{K}_0 = \mathcal{U}_2^{\mathcal{M}} V$, where $V \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ is an isometry ($VV^* = I$).*

Proof Let $\mathcal{R}_0 = \mathcal{K}_0 \ominus Z\mathcal{K}_0$. This is a D-invariant subspace in $\mathcal{U}_2^{\mathcal{N}}$. We can assume it is non-empty, for else $\mathcal{K}_0 = Z\mathcal{K}_0 = Z^n\mathcal{K}_0$ for all $n \geq 0$, and since $X \in \mathcal{U}_2 \Rightarrow \lim_{n\to\infty} \mathbf{P}(Z^{-n}X) = 0$, this implies that $\mathcal{K}_0 = 0$, and there is nothing to prove. Likewise, define $\mathcal{R}_n = Z^n\mathcal{K}_0 \ominus Z^{n+1}\mathcal{K}_0$. Then $\mathcal{R}_n = Z^n\mathcal{R}_0$, and $\mathcal{K}_0 = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots$.

Suppose s-dim $\mathcal{R}_0 = M$, and define the sequence of Hilbert spaces $\mathcal{M} = \cdots \times \mathcal{M}_0 \times \mathcal{M}_1 \times \cdots$ to have entries $\mathcal{M}_k = \mathbb{C}^{M_k}$ ($\mathcal{M}_k = \ell_2$ if $M_k = \infty$). Then there exist isometries $V_k : \mathcal{M}_k \to (\mathcal{R}_0)_k$ such that $(\mathcal{R}_0)_k = \mathcal{M}_k V_k$. Let V be the operator whose k-th block-row is equal to V_k. We obtain an orthonormal basis representation of $\mathcal{R}_0$, as in section 2, such that

$$\mathcal{R}_0 = \mathcal{D}_2^{\mathcal{M}} V, \qquad \mathbf{P}_0(VV^*) = I.$$

Then $\mathcal{R}_n = \mathcal{D}_2 Z^n V$. Because $\mathcal{R}_i \perp \mathcal{R}_j$ ($i \neq j$), it follows that $D_1 Z^n V \perp D_2 V$ ($n \geq 1$) for all $D_{1,2} \in \mathcal{D}_2$, *i.e.,*

$$\begin{array}{rcl} \mathbf{P}_0(Z^n VV^*) & = & 0 \\ \mathbf{P}_0(VV^* Z^{-n}) & = & 0 \end{array}$$

so that $VV^* = I$: V is an isometry. The orthogonal collection $\{\mathcal{D}_2 Z^n V\} \in \mathcal{K}_0$ ($n \geq 0$), and together spans the space $\mathcal{U}_2 V$. Hence $\mathcal{K}_0 = \{\mathcal{D}_2 Z^n V\} = \mathcal{U}_2 V$. The factor V is unique up to a left diagonal unitary factor. □

The above proof is along the lines of the proof of Helson [7, §VI.3] for the time-invariant Hardy space setting. This proof was in turn based on the work of Beurling for the scalar (SISO) case and Lax for the extension to vector valued functions.

Note that, in the above theorem, $\mathcal{M}$ can have components $\mathcal{M}_k$ which are infinite dimensional, even if $\mathcal{N}$ is locally finite, depending on $\mathcal{K}_0$. In our application of the theorem in the next section, however, $\mathcal{K}_0$ is such that $\mathcal{M}$ will be locally finite automatically, starting from locally finite spaces.

Corollary 4. *If $V \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ is an isometry, then there exists an isometry $U \in \mathcal{U}(\mathcal{M}_U, \mathcal{N})$ such that* $\ker(\,\cdot\, V^*|_{\mathcal{U}_2^{\mathcal{N}}}) = \mathcal{U}_2^{\mathcal{M}_U} U$. *The operator*

$$W = \begin{bmatrix} U \\ V \end{bmatrix}$$

is inner, with $\mathcal{H}_0(W) = \overline{\mathcal{H}}_0(V)$ and $\mathcal{U}_2 = \overline{\mathcal{H}}_0(V) \oplus \mathcal{U}_2 U \oplus \mathcal{U}_2 V$.

Proof If V is an isometry, then (proposition 1)

$$\mathcal{U}_2^{\mathcal{N}} = \overline{\mathcal{H}}_0(V) \oplus \ker(\,\cdot\, V^*|_{\mathcal{U}_2^{\mathcal{N}}}) \oplus \mathcal{U}_2^{\mathcal{M}} V, \tag{10}$$

where $\mathcal{K}_0' := \ker(\,\cdot\, V^*|_{\mathcal{U}_2})$ is shift-invariant, so that according to theorem 3 there exists an isometry $U \in \mathcal{U}(\mathcal{M}_U, \mathcal{N})$ such that $\mathcal{K}_0' = \mathcal{U}_2^{\mathcal{M}_U} U$. In view of proposition 1, W is inner if $WW^* = I$ and $\ker(\,\cdot\, W^*|_{\mathcal{U}_2}) = 0$. $WW^* = I$ requires $UV^* = 0$, which is true because $\mathcal{U}_2 V \perp \mathcal{U}_2 U$. Hence $\mathcal{U}_2 W = \mathcal{U}_2 U \oplus \mathcal{U}_2 V$, and since $\overline{\mathcal{H}}_0(W) \supset \overline{\mathcal{H}}_0(V)$, we must have (from equation (10)) that $\overline{\mathcal{H}}_0(W) = \overline{\mathcal{H}}_0(V)$ and $\ker(\,\cdot\, W^*|_{\mathcal{U}_2}) = 0$. Hence W is inner, and $\mathcal{H}_0(W)$ is closed. □

6. Inner-outer factorization

We will say that an operator $T_0 \in \mathcal{U}$ is (left) *outer* if

$$\overline{\mathcal{U}_2 T_0} = \mathcal{U}_2 . \tag{11}$$

Other definitions are possible (see *e.g.,* Arveson [1]); the above definition is such that $\overline{\text{ran}}(\cdot T_0) = \overline{\mathcal{X}_2 T_0} = \mathcal{X}_2$, so that $\ker(\cdot T_0^*) = 0$ and T_0 has an algebraic left inverse which is upper (it can be unbounded if $\mathcal{X}_2 T_0$ is not closed). A factorization of an operator T into $T = T_0 V$, where T_0 is outer and V is inner (or an isometry: $VV^* = I$) is called an outer-inner factorization. This factorization can be obtained from theorem 3 by defining $\mathcal{K}_0$ to be $\mathcal{K}_0 = \overline{\mathcal{U}_2 T}$. The closures here and in (11) are necessary in cases where $\mathcal{U}_2 T$ is not a closed subspace. This happens when there are 'zeros on the unit circle', for example when $T = I - Z$. The existence of inner-outer factorizations is established in the following theorem. A more general proof (in the context of nest algebras which specializes to the current setting) is given by Arveson [1].

Theorem 5. *Let $T \in \mathcal{U}(\mathcal{M},\mathcal{N})$. Then T has a factorization*

$$T = T_0 V$$

where $V \in \mathcal{U}(\mathcal{M}_V,\mathcal{N})$ is an isometry ($VV^* = I$), $T_0 \in \mathcal{U}(\mathcal{M},\mathcal{M}_V)$ is outer, and $\mathcal{M}_V \subset \mathcal{M}$ (entry-wise). V is unique up to a left diagonal unitary factor.

Proof Define $\mathcal{K}_0 = \overline{\mathcal{U}_2 T}$. Then $\mathcal{K}_0$ is a D-invariant subspace which is shift-invariant: $Z\mathcal{K}_0 \subset \mathcal{K}_0$. According to theorem 3, there exists a space sequence $\mathcal{M}_V$ and an isometric operator $V \in \mathcal{U}(\mathcal{M}_V,\mathcal{N})$ such that $\overline{\mathcal{U}_2^{\mathcal{M}} T} = \mathcal{U}_2^{\mathcal{M}_V} V$. By construction, $\overline{\mathcal{U}_2^{\mathcal{M}} T} = \mathcal{D}_2^{\mathcal{M}_V} V \oplus Z\overline{\mathcal{U}_2^{\mathcal{M}} T}$ with $\mathcal{M}_V$ of minimal dimensions. Because also $\overline{\mathcal{U}_2^{\mathcal{M}} T} = \overline{[\mathcal{D}_2^{\mathcal{M}} \oplus Z\mathcal{U}_2^{\mathcal{M}}] T}$, but $\mathcal{D}_2^{\mathcal{M}} T$ is not necessarily orthogonal to $Z\mathcal{U}^{\mathcal{M}} T$, it follows that $\mathcal{M}_V \subset \mathcal{M}$. In particular, the entries of $\mathcal{M}_V$ are finite vector spaces.

Define $T_0 = TV^*$. Then $\overline{\mathcal{U}_2 T_0} = \overline{\mathcal{U}_2 TV^*} = \overline{\overline{\mathcal{U}_2 T} V^*} = \overline{\mathcal{U}_2 VV^*} = \mathcal{U}_2$, so that T_0 is outer. It remains to prove that $T = T_0 V$, *i.e.,* $T = TV^* V$. This is immediate if V is inner. If V is not inner, then corollary 4 ensures the existence of an isometry U such that

$$\mathcal{U}_2 = \overline{\mathcal{H}}_0(V) \oplus \mathcal{U}_2 U \oplus \mathcal{U}_2 V ,$$

where $\mathcal{K}_0' := \mathcal{U}_2 U = \ker(\cdot V^*|_{\mathcal{U}_2})$, and $W = \begin{bmatrix} U \\ V \end{bmatrix}$ is inner. Then $U^*U + V^*V = I$, $VU^* = 0$, and $T = TV^*V \;\Leftrightarrow\; T(I - V^*V) = 0 \;\Leftrightarrow\; TU^*U = 0$. But $\mathcal{U}_2 TU^* \subset \mathcal{U}_2 VU^* = 0$, which implies $TU^* = 0$. Hence $T = T_0 V$.

To show that V is unique, up to a left diagonal unitary factor, suppose that there exists $V_1 \in \mathcal{U}$: $T = T_1 V_1$, where T_1 is outer and $V_1 V_1^* = I$. Then

$$\overline{\text{ran}}(\cdot T) = \overline{\mathcal{X}_2 T_0 V} = \overline{\mathcal{X}_2 V} = \overline{\text{ran}}(\cdot V) = \overline{\text{ran}}(\cdot V_1) ,$$

so that, according to a theorem of Douglas [5], there exists $X \in \mathcal{X}$ such that $V_1 = XV$. Hence $X = V_1 V^*$, $XX^* = I$, and $X^*X = VV_1^* V_1 V^* = VV^* = I$, so that X is unitary. We also have, from $T_1 = TV_1^* = T_0 VV_1^*$, that

$$\mathcal{U}_2 T_0 = \mathcal{U}_2 T_1 V_1 V^* = \mathcal{U}_2 V_1 V^* \subset \mathcal{U}_2$$

so that $X \in \mathcal{U}$. In the same way, $X^* = X^{-1} = V_1 V^* \subset \mathcal{U}$. Hence $X \in \mathcal{D}$. □

One can show that, in theorem 5, V is inner if and only if $\ker(\cdot\, T^*) = 0$. If V is not inner, then the extension W of V in theorem 5 is inner, and such that $\overline{\mathcal{H}}_0(V) = \mathcal{H}_0(W)$, but the resulting factor $T_0 = TW^*$ based on W is not precisely outer according to the definition in (11):

$$\overline{\mathcal{U}_2 T_0} = \overline{\mathcal{U}_2 T}W^* = \mathcal{U}_2 VW^* = \mathcal{U}_2[0 \quad I]$$

so that this T_0 reaches only a subset of $\mathcal{U}_2$ and maps the rest to 0.

The inner-outer factorization is based on the identification of a subspace $\mathcal{K}_0 = \overline{\mathcal{U}_2 T}$ as $\mathcal{K}_0 = \mathcal{U}_2 V$. The complement in $\mathcal{U}_2$ of this space is $\overline{\mathcal{H}}_0(V) \oplus \mathcal{K}_0'$ and is characterized by the elements $X \in \mathcal{U}_2$ satisfying $\mathbf{P}_0(\mathcal{U}_2 TX^*) = 0$, that is, $XT^* \perp \mathcal{U}_2$. Hence

$$\overline{\mathcal{H}}_0(V) \oplus \mathcal{K}_0' = \{X \in \mathcal{U}_2 : XT^* \in \mathcal{L}_2 Z^{-1}\} = \{X \in \mathcal{U}_2 : \mathbf{P}(XT^*) = 0\}\,.$$

In this expression, $\mathcal{K}_0' = \mathcal{U}_2 U = \ker(\cdot\, V^*|_{\mathcal{U}_2})$ according to its definition. We now show that also $\mathcal{K}_0' = \ker(\cdot\, T^*|_{\mathcal{U}_2}) = \{X \in \mathcal{U}_2 : XT^* = 0\}$. Indeed, if $X \in \mathcal{K}_0'$, then $X = X_1 U$ for some $X_1 \in \mathcal{U}_2$, and because $UT^* = 0$, it follows that $XT^* = 0$. Conversely, if $XT^* = 0$, then $XV^* T_0^* = 0$, and because $\ker(\cdot\, T_0^*) = 0$, it follows that $XV^* = 0$ so that $X \in \mathcal{K}_0'$. Hence $\mathcal{K}_0' = \ker(\cdot\, T^*|_{\mathcal{U}_2})$.

7. Computation of the inner-outer factorization $T = VT_0$

Let $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$, with $\mathcal{M}, \mathcal{N}$ locally finite spaces of sequences. In this section, it will be convenient to work with a dual factorization of T: $T = VT_0$ (for different V and T_0), where T_0 is 'right outer': $\overline{\mathcal{L}_2 Z^{-1} T_0^*} = \mathcal{L}_2 Z^{-1}$ (or $\overline{T_0 \mathcal{U}_2} = \mathcal{U}_2$), and where the left inner (isometric) factor V satisfies $V^*V = I$ and is obtained by identifying the subspace $\mathcal{L}_2 Z^{-1} V^*$ with $\overline{\mathcal{L}_2 Z^{-1} T^*}$. Dual to the other properties mentioned in the previous section, there exists an inner operator $W = [U \quad V]$ that extends V in case it is isometric, and U satisfies $U^*U = I$, $U^*V = 0$, and $\mathcal{L}_2 Z^{-1} U^* = \ker(\cdot\, T|_{\mathcal{L}_2 Z^{-1}})$, so that $U^*T = 0$. For this decomposition,

$$\begin{aligned} \mathcal{L}_2 Z^{-1} &= \overline{\mathcal{H}}(V) \oplus \mathcal{L}_2 Z^{-1} V^* \oplus \mathcal{L}_2 Z^{-1} U^* \\ &= \overline{\mathcal{H}}(V) \oplus \overline{\mathcal{L}_2 Z^{-1} T^*} \oplus \ker(\cdot\, T|_{\mathcal{L}_2 Z^{-1}}) \end{aligned} \tag{12}$$

Define $\mathcal{K}' = \ker(\cdot\, T|_{\mathcal{L}_2 Z^{-1}})$. We have defined in section 3 the decomposition of T, restricted to $\mathcal{L}_2 Z^{-1}$, as

$$\cdot\, T|_{\mathcal{L}_2 Z^{-1}} = \cdot\, K_T + \cdot\, H_T\,, \qquad \cdot\, K_T = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\cdot\, T|_{\mathcal{L}_2 Z^{-1}})\,.$$

It is thus seen that $\overline{\mathcal{H}}(V)$ is the *largest* subspace in $\mathcal{L}_2 Z^{-1}$ for which $\overline{\mathcal{H}}(V) K_T = 0$ and which is orthogonal to $\mathcal{K}'$. U in turn is defined by $U^*U = I$, $\overline{\mathcal{H}}(U) \subset \overline{\mathcal{H}}(V)$ and the condition that $\mathcal{L}_2 Z^{-1} U^*$ is the largest subspace in $\mathcal{L}_2 Z^{-1}$ such that $(\mathcal{L}_2 Z^{-1} U^*) T = 0$. Finally, V is determined by its input state space $\overline{\mathcal{H}}(V)$ and the fact that it complements U. These properties provide a way to compute the inner-outer factorization, first in terms of basis representations of the state spaces of T and V, and then in terms of state space matrices.

Lemma 6. *Let $T \in \mathcal{U}$ be a locally finite transfer operator, and let $\mathbf{F}_0$ be a basis representation of a subspace in $\mathcal{U}_2$ which contains the output state space of T, i.e., such that $\mathcal{H}_0(T) \subset \mathcal{D}_2 \mathbf{F}_0 \subset \mathcal{U}_2$. Let $T = VT_0$ be an inner-outer factorization of T and suppose that $\mathbf{Q}$ is an orthonormal basis representation of the input state space $\overline{\mathcal{H}}(V)$ of V.*

Then $\overline{\mathcal{H}}(V) = \mathcal{D}_2 \mathbf{Q}$ is the largest subspace in $\mathcal{L}_2 Z^{-1}$ for which $\mathbf{Q}T = Y\mathbf{F}_0$ with $\ker(\cdot\, Y) = 0$.

Proof We use the property that $\overline{\mathcal{H}}(V)$ is the largest subspace in $\mathcal{L}_2 Z^{-1}$ for which $\overline{\mathcal{H}}(V)K_T = 0$ and which is orthogonal to $\mathcal{K}'$. The fact that $\overline{\mathcal{H}}(V)K_T = 0$ translates to the condition $\mathbf{Q}T \in \mathcal{U}$. Hence $\overline{\mathcal{H}}(V)T \subset \overline{\mathcal{H}}_0(T)$, and we must have that $\mathbf{Q}T = Y\mathbf{F}_0$ for some bounded diagonal operator Y, which plays an instrumental role in the derivation of a state realization for V. It remains to implement the condition $\overline{\mathcal{H}}(V) \perp \mathcal{K}'$. Suppose that $\mathbf{Q}$ has a component in $\mathcal{K}'$, so that $D\mathbf{Q} \in \mathcal{K}'$, for some $D \in \mathcal{D}_2$. Then, since $\mathcal{K}' = \ker(\,\cdot\, T|_{\mathcal{L}_2 Z^{-1}})$,

$$D\mathbf{Q} \in \mathcal{K}' \quad \Leftrightarrow \quad D\mathbf{Q}T = DY\mathbf{F}_0 = 0 \quad \Leftrightarrow \quad D \in \ker(\,\cdot\, Y)\,.$$

Hence $\overline{\mathcal{H}}(V)$ can be described as the largest subspace $\mathcal{D}_2\mathbf{Q}$ in $\mathcal{L}_2 Z^{-1}$ for which $\mathbf{Q}T = Y\mathbf{F}_0$ with $\ker(\,\cdot\, Y) = 0$. □

If $\mathcal{B}$ is the state sequence space of T, and $\mathcal{B}_V$ is the state sequence space of V, then $Y \in \mathcal{D}(\mathcal{B}_V, \mathcal{B})$. The condition $\ker(\,\cdot\, Y) = 0$ implies that $\mathcal{B}_V \subset \mathcal{B}$ (pointwise), so that the state dimension of V is at each point in time less than or equal to the state dimension of T at that point.

Proposition 7. *Let $T \in \mathcal{U}$ be a locally finite transfer operator, let $\mathbf{T} = \{A, B, C, D\}$ be an observable realization of T, and assume $\ell_A < 1$. Let V be a left inner (isometric) factor of T so that $T_0 = V^*T$ is right outer. Then the pair (A_V, B_V) that corresponds to an orthonormal basis representation $\mathbf{Q}$ of $\overline{\mathcal{H}}(V)$ satisfies*

$$\begin{array}{lrcl} (i) & A_V^* Y A + B_V^* B & = & Y^{(-1)} \\ (ii) & A_V^* Y C + B_V^* D & = & 0 \\ (iii) & A_V^* A_V + B_V^* B_V & = & I \\ (iv) & \ker(\,\cdot\, Y) & = & 0\,, \end{array}$$

for some bounded $Y \in \mathcal{D}$. Conversely, all solutions (A_V, B_V) of these equations give basis representations of subspaces in $\overline{\mathcal{H}}(V)$, and the solution of maximal possible dimensions is a basis representation of $\overline{\mathcal{H}}(V)$. For such a solution, Y is unique save for a left diagonal unitary factor.

Proof Let $\mathbf{F}_0 = (I - AZ)^{-1}C$. Because $\mathbf{F}_0$ is a basis representation of a subspace containing the output state space of T, *i.e.*, $\mathcal{H}_0(T) \subset \mathcal{D}_2\mathbf{F}_0$, we have $\mathbf{P}(\mathbf{Q}T) = Y\mathbf{F}_0$ for some bounded $Y \in \mathcal{D}$, and we will show that Y is given by a solution to equation (i). Indeed, let Y be defined by $\mathbf{P}(\mathbf{Q}T) = Y\mathbf{F}_0$. We will apply the relations $Z\mathbf{Q} = A_V^*\mathbf{Q} + B_V^*$; $\mathbf{F}_0 = C + AZ\mathbf{F}_0$, $T = D + BZ\mathbf{F}_0$ (*cf.* equation (7)). Firstly, $\mathbf{P}(Z^{-1}Y\mathbf{F}_0) = Y^{(1)}\mathbf{P}(Z^{-1}\mathbf{F}_0) = (YA)^{(1)}\mathbf{F}_0$. On the other hand,

$$\begin{array}{rcl} A_V^{*(1)}\mathbf{P}(Z^{-1}\mathbf{Q}T) & = & \mathbf{P}(Z^{-1}[A_V^*\mathbf{Q}]T) \\ & = & \mathbf{P}(Z^{-1}[Z\mathbf{Q} - B_V^*]T) \\ & = & \mathbf{P}(\mathbf{Q}T) - B_V^{*(1)}B^{(1)}\mathbf{F}_0 \\ & = & Y\mathbf{F}_0 - B_V^{*(1)}B^{(1)}\mathbf{F}_0\,. \end{array}$$

Hence, because observability means that $\cdot\,\mathbf{F}_0|_{\mathcal{D}_2}$ is one-to-one,

$$\begin{array}{l} \mathbf{P}(Z^{-1}Y\mathbf{F}_0) = \mathbf{P}(Z^{-1}\mathbf{Q}T) \\ \Rightarrow \quad (A_V^*YA)^{(1)}\mathbf{F}_0 + (B_V^*B)^{(1)}\mathbf{F}_0 = Y\mathbf{F}_0 \\ \Leftrightarrow \quad A_V^*YA + B_V^*B = Y^{(-1)}\,. \end{array}$$

Conversely, since $\ell_A < 1$ implies that any solution Y of (i) is bounded and unique (given (A_V, B_V)), it follows that a basis representation $\mathbf{Q}$ associated to a given pair (A_V, B_V) satisfying (i) will satisfy $\mathbf{P}(\mathbf{Q}T) = Y\mathbf{F}_0$.

Let Y be given by $\mathbf{P}(\mathbf{Q}T) = Y\mathbf{F}_0$. To derive the equivalence of (*ii*) with the condition $\mathbf{Q}T = \mathcal{U}$, we will use the fact that $\mathbf{Q}T \in \mathcal{U} \Leftrightarrow \mathbf{P}_0(Z^n\mathbf{Q}T) = 0$ for all $n > 0$.

$$\begin{array}{rrcl} n = 1: & \mathbf{P}_0(Z\mathbf{Q}T) & = & \mathbf{P}_0(\,[A_V^*\mathbf{Q} + B_V^*]T) \\ & & = & A_V^*\mathbf{P}_0(\mathbf{Q}T) + B_V^*D \\ & & = & A_V^*YC + B_V^*D\,. \end{array}$$

Hence $\mathbf{P}_0(Z\mathbf{Q}T) = 0 \;\Leftrightarrow\; A_V^*YC + B_V^*D = 0$. For $n > 1$, assume $\mathbf{P}_0(Z^{n-1}\mathbf{Q}T) = 0$. Then

$$\begin{array}{rcl} \mathbf{P}_0(Z^n\mathbf{Q}T) & = & \mathbf{P}_0(Z^{n-1}[Z\mathbf{Q}T]) \\ & = & \mathbf{P}_0(Z^{n-1}[A_V^*\mathbf{Q}]T) \;+\; \mathbf{P}_0(Z^{n-1}B_V^*T) \\ & = & A_V^{*(n-1)}\mathbf{P}_0(Z^{n-1}\mathbf{Q}T) \;+\; B_V^{*(n-1)}\mathbf{P}_0(Z^{n-1}T) \\ & = & 0 \;+\; 0\,. \end{array}$$

Hence (*ii*) is both necessary and sufficient for the condition $\mathbf{Q}T \in \mathcal{U}$ to be satisfied. The fact that we took $\mathbf{Q}$ to be an orthonormal basis representation implies condition (*iii*), and condition (*iv*) has already been derived.

Conversely, if (A_V, B_V) is a solution of (*i*)–(*iv*), then we have already shown that there is a unique Y satisfying (*i*), and that (*i*) and (*ii*) imply $\mathbf{Q}T = Y\mathbf{F}_0$. Since, by (*iv*), also $\ker(\,\cdot\,Y) = 0$, lemma 6 yields $\mathcal{D}_2\mathbf{Q} \subset \overline{\mathcal{H}}(V)$, as $\overline{\mathcal{H}}(V)$ is the largest subspace satisfying these conditions, and $\mathbf{Q}$ must be part of a basis representation of $\overline{\mathcal{H}}(V)$. The existence of the inner factor implies that a solution of maximal dimensions of these equations yields a basis representation of $\overline{\mathcal{H}}(V)$.

To prove the uniqueness of Y save for a left diagonal unitary factor, let the triple (A_V', B_V', Y') be another solution of (*i*)–(*iv*) generating $\overline{\mathcal{H}}(V)$. (A_V', B_V') generates another orthonormal basis representation $\mathbf{Q}'$, but two orthonormal basis representations of the same subspace are related by a unitary transformation. It follows that the second solution must be related to the first one via a unitary state transformation R: $(A_V', B_V', Y') = (RA_VR^{-(-1)}, B_VR^{-(-1)}, RY)$. Hence Y is unique up to a left unitary diagonal factor. □

It is possible to construct solutions (A_V, B_V) for the four equations in proposition 7, and from these solutions a realization $\mathbf{V}$ for the inner (isometric) factor V of T follows. Taking the k-th entry of each diagonal in (*i*)–(*iv*) gives the recursive matrix equations

$$\left\{ \begin{array}{lrcl} (i) & A_{V,k}^*\,Y_k\,A_k \;+\; B_{V,k}^*B_k & = & Y_{k+1} \\ (ii) & A_{V,k}^*\,Y_k\,C_k \;+\; B_{V,k}^*D_k & = & 0 \\ (iii) & A_{V,k}^*A_{V,k} \;+\; B_{V,k}^*B_{V,k} & = & I \\ (iv) & Y_{k+1} \text{ full row-rank.} & & \end{array} \right.$$

$A_{V,k}$ and $B_{V,k}$ can be computed from these equations starting at some point in time, once an initial value for Y is known (this is discussed below). Assuming Y_k known, the computation of Y_{k+1}, $A_{V,k}$ and $B_{V,k}$ can be done as discussed in the following theorem. It will be convenient to compute $W = [U \;\; V]$, *i.e.,* to compute U along with V.

Theorem 8. *Let $T \in \mathcal{U}$ have an observable realization $\mathbf{T} = \{A, B, C, D\}$ with $\ell_A < 1$. Then there exists a unitary block diagonal operator $\mathbf{W}$ with entries denoted by*

$$\mathbf{W} = \begin{bmatrix} A_V & C_U & C_V \\ B_V & D_U & D_V \end{bmatrix} \tag{13}$$

and a diagonal operator $Y \in \mathcal{D}$, satisfying the equation

$$\mathbf{W}^* \begin{bmatrix} Y & \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} Y^{(-1)} & 0 \\ 0 & 0 \\ B_{T_0} & D_{T_0} \end{bmatrix}, \qquad \begin{array}{l} \ker(\cdot\, Y) = 0 \\ \ker(\cdot\, D_{T_0}) = 0\,. \end{array} \tag{14}$$

for some bounded B_{T_0} and D_{T_0}, such that the resulting Y is of maximal possible dimensions. Y is unique up to a left diagonal unitary factor. **W** *is a minimal realization of an inner operator $W = [U \;\; V]$ where V is the factor of an inner-outer factorization of T, $T_0 = V^*T$ is outer and $U^*T = 0$. Realizations of V and T_0 are*

$$\mathbf{V} = \begin{bmatrix} A_V & C_V \\ B_V & D_V \end{bmatrix}, \qquad \mathbf{T}_0 = \begin{bmatrix} A & C \\ B_{T_0} & D_{T_0} \end{bmatrix}.$$

(The realization of V is minimal, but the realization of T_0 is not necessarily minimal.)

Proof The equations which **W** and Y have to satisfy are, written in full,

$$\begin{array}{llcl} (a) & A_V^* YA + B_V^* B & = & Y^{(-1)} \\ (b) & A_V^* YC + B_V^* D & = & 0 \\ (c) & \ker(\cdot\, Y) & = & 0 \end{array} \qquad \begin{array}{llcl} (d) & C_U^* YA + D_U^* B & = & 0 \\ (e) & C_U^* YC + D_U^* D & = & 0 \end{array} \qquad \begin{array}{llcl} (f) & C_V^* YA + D_V^* B & = & B_{T_0} \\ (g) & C_V^* YC + D_V^* D & = & D_{T_0} \\ (h) & \ker(\cdot\, D_{T_0}) & = & 0\,. \end{array}$$

Let V be the inner (isometric) factor of the inner-outer factorization of T, and let $W = [U \;\; V]$ be the extension of V to an inner operator. The existence of V and W has been proven in theorem 5. Let **W** be a minimal unitary realization of W, and denote the entries of **W** as in equation (13). The existence claim of the theorem is that this **W** satisfies equations (a)–(h) above. Indeed, (A_V, B_V) satisfy equations (i)–(iv) in proposition 7, for some bounded Y. These correspond to (a)–(c).

In addition, U satisfies $U^*T = 0$. In order to evaluate this product, denote the orthonormal basis representation of $\overline{\mathcal{H}}(V)$ corresponding to (A_V, B_V) by $\mathbf{Q}$, so that $U = D_U + \mathbf{Q}^* C_U$ and $V = D_V + \mathbf{Q}^* C_V$. Let $\mathbf{F}_0 = (I - AZ)^{-1} C$, so that $T = D + BZ\mathbf{F}_0$. U^*T evaluates as

$$\begin{array}{rcl} U^*T & = & [D_U^* + C_U^*\mathbf{Q}]T \\ & = & D_U^*[D + BZ\,\mathbf{F}_0] + C_U^* Y\,\mathbf{F}_0 \\ & = & [D_U^* D + C_U^* YC] + [D_U^* B + C_U^* YA] Z\,\mathbf{F}_0\,. \end{array} \tag{15}$$

Hence, as $\mathbf{F}_0$ is one-to-one because the realization of T is observable,

$$U^*T = 0 \qquad \Leftrightarrow \qquad D_U^* D + C_U^* YC = 0 \quad \text{and} \quad D_U^* B + C_U^* YA = 0\,. \tag{16}$$

Thus, equations (d)–(e) are satisfied. Evaluating $T_0 = V^*T$ in terms of state space quantities yields, much as in equation (15),

$$T_0 \;=\; [C_V^* YC + D_V^* D] \;+\; [C_V^* YA + D_V^* B] Z\,\mathbf{F}_0$$

$$\mathbf{T}_0 \;=\; \begin{bmatrix} A & C \\ C_V^* YA + D_V^* B & C_V^* YC + D_V^* D \end{bmatrix}. \tag{17}$$

Hence, B_{T_0} and D_{T_0} satisfy $B_{T_0} = C_V^* YA + D_V^* B$ and $D_{T_0} = C_V^* YC + D_V^* D$, which corresponds to ($f$) and ($g$). Because T_0 is outer, D_{T_0} satisfies (h).

It remains to prove that any such solution **W** leads to an inner-outer factorization. With the partitioning of **W** as indicated and **W** satisfying equations (a)–(h), it is directly verified that A_V and B_V are operators of maximal possible dimensions satisfying equations (i)–(iv), so that the corresponding input state space $\overline{\mathcal{H}}(V)$ is the correct input state space of the inner factor.

From conditions (d) and (e) and equation (16), it follows that the operator U specified by the realization $\{A_V, B_U, C_V, D_U\}$ satisfies $U^*T = 0$. Moreover, U is the operator of maximal possible dimensions satisfying these conditions, since the zero entries at the right-hand side of (14) are of maximal possible dimensions, and hence $\mathcal{L}_2 Z^{-1} U^* = \ker(\cdot\, T|_{\mathcal{L}_2 Z^{-1}})$.

Because $\mathbf{W}$ is a unitary realization, the corresponding operator $W = [U \;\; V]$ is inner, and V has realization $\mathbf{V}$. It follows from equation (12) that V satisfies $\mathcal{L}_2 Z^{-1} V^* = \mathcal{L}_2 Z^{-1} T^*$, so that V is indeed the inner factor of the inner-outer factorization of T. With this realization of the inner factor and B_{T_0} and D_{T_0} satisfying (f)–(h), $\mathbf{T}_0$ is a realization for the outer factor T_0. □

The above theorem can be used to actually compute the inner-outer factorization. Indeed, upon taking the k-th entry along each diagonal of equation (14), we obtain the recursion: given Y_k, compute $\mathbf{W}_k$ such that

$$\mathbf{W}_k^* \begin{bmatrix} Y_k & \\ & I \end{bmatrix} \begin{bmatrix} A_k & C_k \\ B_k & D_k \end{bmatrix} = \begin{bmatrix} Y_{k+1} & 0 \\ 0 & 0 \\ (B_{T_0})_{k+1} & (D_{T_0})_{k+1} \end{bmatrix} \qquad \begin{array}{l} \ker(\cdot\, Y_{k+1}) = 0 \\ \ker(\cdot\, (D_{T_0})_{k+1}) = 0\,. \end{array}$$

$\mathbf{W}_k$ can be obtained by a simple QR-factorization. The resulting algorithm is familiar from (non-stationary) Kalman filtering, where it is known as the square-root algorithm.

One issue that remains to be discussed concerns the initialization of Y. If the input space $\mathcal{M}$ for T (and hence V) has empty dimensions before time instant 1, or the state dimension of T is zero at time instant 1, then an initial point is $Y_1 = [\,\cdot\,]$, so that a minimal realization for V starts with zero states at time instant 1. For the more general class of systems which are time-invariant before, say, point 1 in time, an initial value for Y is determined in the following way. Y_1 now has to satisfy an equation rather than a recursion: $Y_1 = Y_0 = A_{V,0}^* Y_0 A_0 + B_{V,0}^* B_0$, where, as before,

$$\begin{aligned} A_{V,0}^* A_{V,0} + B_{V,0}^* B_{V,0} &= I\,, \\ A_{V,0}^* Y_0 C_0 + B_{V,0}^* D_0 &= 0\,. \end{aligned}$$

We will show that the solution of these equations is the same as the classical solution of the inner-outer factorization, and is determined by the transmission zeros of the time-invariant part of T that are located in the unit disc. For convenience of notation, define $y = Y_0$, $a = A_0$, $b = B_0$, $c = C_0$, $d = D_0$, $\alpha = A_{V,0}$, $\beta = B_{V,0}$. We will also assume that d (and hence T) is invertible, and that its zeros are distinct. Then

$$\begin{array}{rcl} y &=& \alpha^* y a + \beta^* b \\ 0 &=& \alpha^* y c + \beta^* d \\ I &=& \alpha^* \alpha + \beta^* \beta \end{array} \quad \Leftrightarrow \quad \begin{array}{rcl} \beta^* &=& -\alpha^* y\, c d^{-1} \\ y &=& \alpha^* y\, (a - c d^{-1} b) \\ I &=& \alpha^* \alpha + \beta^* \beta . \end{array} \tag{18}$$

Bring in eigenvalue decompositions of α and $(a - cd^{-1}b)$: $\alpha = r\phi r^{-1}$; $a - cd^{-1}b = s\psi s^{-1}$. Then

$$(r^* y s) = \phi^* (r^* y s) \psi .$$

Because both ϕ and ψ are diagonal matrices, the above expression shows that $(r^* ys)$ must be a rectangular diagonal matrix (or a permutation thereof), and hence the diagonal entries of ϕ are equal to a subset of the diagonal entries of ψ^{-*}. In view of the requirement $\alpha^*\alpha = I - \beta^*\beta$, ϕ can contain only the entries of ψ^{-*} that are smaller than 1. Because V must be of the highest possible system order while y must have full row rank, ϕ is precisely equal to those entries. It remains to note that the entries of $\psi^{-1} = \operatorname{eig}(a - cd^{-1}b)^{-1}$ are equal to the transmission zeros of T. This is because $T^{-1} = d^{-1} + d^{-1}bz\,\left[I - (a - cd^{-1}b)z\right]^{-1} cd^{-1}$ has poles equal to $\operatorname{eig}(a - cd^{-1}b)^{-1}$. With the poles of the inner system thus determined, it is a straightforward matter (involving a Lyapunov equation) to compute α, β, and y from (18).

8. Closed-form expression for the outer factor realization

In the time-invariant setting, it is well known that the outer factor T_0 of T can be written in closed form in terms of the original state matrices $\{A, B, C, D\}$ of T and only one unknown intermediate quantity, which is the solution of a Riccati equation with $\{A, B, C, D\}$ as parameters. We will show in this section how a Riccati recursive equation can be derived from the square-root algorithm in theorem 8. Denote by $(\,\cdot\,)^\dagger$ the pseudo-inverse of an operator.

Theorem 9. *Let $T \in \mathcal{U}$ be a locally finite transfer operator, let $\mathbf{T} = \{A, B, C, D\}$ be an observable realization of T, and assume $\ell_A < 1$.*

There exist bounded solutions $M \geq 0$ of

$$M^{(-1)} = A^*MA + B^*B - \left[A^*MC + B^*D\right] (D^*D + C^*MC)^\dagger \left[D^*B + C^*MA\right] , \tag{19}$$

and there is a unique bounded solution of maximal rank. Let M be this solution, and let R be a minimal full range factor ($\ker(\,\cdot R^*) = 0$*) of*

$$RR^* = (D^*D + C^*MC)^\dagger \,. \tag{20}$$

A realization of the outer factor T_0 of T so that $T_0 = V^*T$ is then given by

$$\mathbf{T}_0 = \begin{bmatrix} I & \\ & R^* \end{bmatrix} \begin{bmatrix} A & C \\ C^*MA + D^*B & C^*MC + D^*D \end{bmatrix} . \tag{21}$$

Proof We will prove the equivalence of equations (19), (20) and (21) with (14).

Taking the square of equation (14) and using the fact that $\mathbf{W}$ is unitary leads to the set of equations

$$\begin{aligned} A^*MA + B^*B &= M^{(-1)} + B_{T_0}^*B_{T_0} \\ C^*MC + D^*D &= D_{T_0}^*D_{T_0}\,, \qquad \ker(\,\cdot\, D_{T_0}) = 0 \\ C^*MA + D^*B &= D_{T_0}^*B_{T_0} \end{aligned} \tag{22}$$

where we have put $M := Y^*Y$. Because these equations are equivalent to (14), there is a solution $M \geq 0$ of maximal rank, corresponding to Y of maximal dimensions with $\ker(\,\cdot\, Y) = 0$. This solution M is unique, as Y is unique up to a left diagonal unitary factor (proposition 7).

Let M be this solution. Operators D_{T_0} and B_{T_0} that, with M, satisfy (22) are obtained as follows. The second equation gives $D_{T_0} = \Phi(D^*D + C^*MC)^{1/2}$, for some isometry Φ such that $\ker(\,\cdot\, D_{T_0}) = 0$. Let $R = D_{T_0}^\dagger$ denote the generalized inverse of D_{T_0}. It satisfies equation (20), and also $D_{T_0} = R^\dagger = R^*(RR^*)^\dagger = R^*(D^*D + C^*MC)$, but R need not be a bounded operator unless the range of D_{T_0} is closed. However, we know at this point that there is a bounded solution B_{T_0} such that $D_{T_0}^*B_{T_0} = (C^*MA + D^*B)$, and, according to a theorem of Douglas [5], we can take $B_{T_0} = (D_{T_0}^*)^\dagger(C^*MA + D^*B) = R^*(C^*MA + D^*B)$, which is well defined and bounded. Substitution of this solution in the first equation shows that M satisfies the Riccati equation (19).

Conversely, any solution $M \ge 0$ satisfies, with $D_{T_0} = \Phi(D^*D+C^*MC)^{1/2}$ and $B_{T_0} = (D^*_{T_0})^{\dagger}(C^*MA+D^*B)$, equations (22), so that a solution of maximal rank of the Riccati equation is also a solution of maximal rank of (22). Hence, the Riccati equation has a unique solution M of maximal rank, and this solution leads to a realization of an outer factor T_0 via (20) and (21). □

The Riccati equation has more solutions M. For example, if D^*D is invertible, then $M = 0$ is always a solution, and yields $T_0 = T$. If the rank of M is not maximal, then T_0 will not be outer.

Initial conditions for M can be obtained as $M_{k_0} = [\,\cdot\,]$ when T starts with zero states at some point k_0 in time, or from a solution of the Riccati equation if T is time-invariant before k_0. Again, the solution requires eigenvalue decompositions, and must satisfy the side conditions that $M \ge 0$ and has maximal rank. Initial conditions for the related spectral factorization problem are investigated in [12]. For the Riccati equation occurring in the spectral factorization problem, it has been proven that there is a unique solution $M \ge 0$, and that an approximate initial point $\hat{M}_{k_0} = 0$ of the recursion will strongly converge to the correct solution $M \ge 0$. This result does not apply to the present context: it is possible that $M = 0$ is a solution which does not correspond to a factor which is outer.

9. Inner-outer factorization examples

We finish this paper with some example results of the inner-outer factorization algorithm. In example 2–4, we consider finite (4×4) matrices, as a special case of time varying systems which have input and output spaces that are zero except for a finite time-interval. In this case, interesting things can occur only when T is singular or when the dimensions of T are not uniform.

1. Consider the time-invariant system

$$T = \frac{z-\alpha^*}{1-\beta z} = \frac{1-\alpha z}{1-\beta z} \cdot \frac{z-\alpha^*}{1-\alpha z},$$

   $(|\alpha|,\ |\beta| < 1)$. A state-space realization of T is

$$\mathbf{T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1-\alpha^*\beta & -\alpha^* \end{bmatrix}.$$

   Its zeros are $(a - cd^{-1}b)^{-1} = (\beta - \alpha^{-*}(1-\alpha^*\beta))^{-1} = \alpha^*$. Hence α is indeed a solution of equation (18). Substitution leads to $\beta = (1-\alpha^*\alpha)^{1/2}$ and $y = \beta^*$.

2. The algorithm, applied to

$$T = \begin{bmatrix} \underline{0} & 1 & 4 & 6 \\ 0 & \underline{0} & 2 & 5 \\ 0 & 0 & \underline{0} & 3 \\ 0 & 0 & 0 & \underline{0} \end{bmatrix}$$

   (the underlined entries form the 0-th diagonal) yields an almost trivial isometric factor V or inner factor W (the dots correspond to columns or rows with vanishing dimensions, and $\#\mathcal{M}$

denotes the sequence of dimensions of the space $\mathcal{M}$):

$$V = \begin{bmatrix} \underline{\vdots} & 1 & 0 & 0 \\ \cdot & \underline{0} & 1 & 0 \\ \cdot & 0 & \underline{0} & 1 \\ \cdot & 0 & 0 & \underline{0} \end{bmatrix} \qquad W = \begin{bmatrix} \underline{\vdots} & 1 & 0 & 0 & 0 \\ \cdot & \underline{0} & 1 & 0 & 0 \\ \cdot & 0 & \underline{0} & 1 & 0 \\ \cdot & 0 & 0 & \underline{0} & \underline{1} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_W & = & [1\ 1\ 1\ 1] \\ \#\mathcal{N}_W & = & [0\ 1\ 1\ 2] \end{array}$$

It is seen that V is not inner, because T is singular. W is the inner extension of V. The only effect of W is a redefinition of time intervals: W acts as a shift operator. $T_0 = W^*T$ is

$$W^*T = \begin{bmatrix} \underline{\vdots} & \cdot & \cdot & \cdot \\ 0 & \underline{1} & 4 & 6 \\ 0 & 0 & \underline{2} & 5 \\ 0 & 0 & 0 & \underline{3} \\ 0 & 0 & 0 & \underline{0} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_{T_0} & = & [0\ 1\ 1\ 2] \\ \#\mathcal{N}_{T_0} & = & [1\ 1\ 1\ 1] \end{array}$$

The multiplication by W^* has shifted the rows of T downwards. This is possible: the result T_0 is still upper. V^*T is equal to W^*T with its last row removed.

3. Take

$$T = \begin{bmatrix} \underline{0} & 1 & 4 & 6 \\ 0 & \underline{1} & 2 & 5 \\ 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M} & = & [1\ 1\ 1\ 1] \\ \#\mathcal{N} & = & [1\ 1\ 1\ 1] \\ \#\mathcal{B} & = & [0\ 1\ 2\ 1] \end{array}$$

$\mathcal{B}$ is the state sequence space of T. T is again singular, but now a simple shift will not suffice. The algorithm computes W as

$$W = \begin{bmatrix} \underline{\vdots} & -0.707 & 0.577 & 0.367 & 0.180 \\ \cdot & \underline{-0.707} & -0.577 & -0.367 & -0.180 \\ \cdot & 0 & \underline{0.577} & -0.733 & -0.359 \\ \cdot & 0 & 0 & \underline{-0.440} & \underline{0.898} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_W & = & [1\ 1\ 1\ 1] \\ \#\mathcal{N}_W & = & [0\ 1\ 1\ 2] \\ \#\mathcal{B}_W & = & [0\ 1\ 1\ 1] \end{array}$$

$$T_0 = W^*T = \begin{bmatrix} \underline{\vdots} & \cdot & \cdot & \cdot \\ 0 & \underline{-1.414} & -4.243 & -7.778 \\ 0 & 0 & \underline{1.732} & 2.309 \\ 0 & 0 & 0 & \underline{-2.273} \\ 0 & 0 & 0 & \underline{0} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_{T_0} & = & [0\ 1\ 1\ 2] \\ \#\mathcal{N}_{T_0} & = & [1\ 1\ 1\ 1] \end{array}$$

V is equal to W with its last column removed, so that $T_0 = V^*T$ is equal to the above T_0 with its last row removed. It is seen that the state dimension of W and V is smaller than that of T.

4. In the previous examples, we considered systems T with a constant number of inputs and outputs (equal to 1), for which $V \neq I$ only if T is singular. However, a non-identical V can also occur if the number of inputs and outputs of T are varying in time. Thus consider

$$T = \begin{bmatrix} \underline{1.000} & 0.500 & 0.250 & 0.125 \\ \underline{1.000} & 0.300 & 0.100 & 0.027 \\ 0 & \underline{1.000} & 0.500 & 0.250 \\ 0 & 0 & \underline{1.000} & 0.300 \\ \cdot & \cdot & \cdot & \underline{\vdots} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M} & = & [2\ 1\ 1\ 0] \\ \#\mathcal{N} & = & [1\ 1\ 1\ 1] \\ \#\mathcal{B} & = & [0\ 1\ 2\ 1] \end{array}$$

$$V = \begin{bmatrix} \underline{-0.707} & 0.099 & 0.025 & -0.699 \\ \underline{-0.707} & -0.099 & -0.025 & 0.699 \\ 0 & \underline{0.990} & -0.005 & 0.139 \\ 0 & 0 & \underline{0.999} & 0.035 \\ \cdot & \cdot & \cdot & \underline{\cdot} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_V & = & [2\ 1\ 1\ 0] \\ \#\mathcal{N}_V & = & [1\ 1\ 1\ 1] \\ \#\mathcal{B}_V & = & [0\ 1\ 1\ 1] \end{array}$$

In this case, V is itself inner. The outer factor T_0 follows as

$$T_0 = V^* T = \begin{bmatrix} \underline{-1.414} & -0.565 & -0.247 & -0.107 \\ 0 & \underline{1.010} & 0.509 & 0.257 \\ 0 & 0 & \underline{1.001} & 0.301 \\ 0 & 0 & 0 & \underline{-0.023} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_{T_0} & = & [1\ 1\ 1\ 1] \\ \#\mathcal{N}_{T_0} & = & [1\ 1\ 1\ 1] \end{array}$$

An interesting observation from these examples is that the inner-outer factorization of finite matrices T is equal to the QR-factorization of T when it is considered as an ordinary matrix without block entries.

10. Concluding remarks

We have derived a simple algorithm to compute realizations of the (left) inner and outer factors of a realization of a given system T. The computations are unidirectional: starting from an initial value of a quantity Y_k, state matrices are computed recursively from that point on. The initial value can be obtained straightforwardly in cases where the state dimension of T vanishes before some point in time, or where T is time-invariant before a point in time. From the algorithm, it can be observed that the number of states in the inner factor (the number of 'zeros inside the unit disc') is at each point k always less than the number of states of T, and cannot change at point k if D_k is square and invertible at that point, unless the number of states of T decreases at that point. It can increase if D_k is singular, or if the number of inputs increases at that point, and can decrease if the state dimension of T decreases, or if the number of inputs decreases. For finite matrices, the inner-outer factorization reduces to a QR-factorization.

The outer factor can be computed as a by-product of the same algorithm, or alternatively via a Riccati-type recursive equation.

Additional research is required to answer the following questions: (1) does there exist an approximate initial point to start the Riccati recursion, such that it will converge to the exact solution for large k, and (2) can the observability condition be lifted. For the related Riccati recursion that occurs in the spectral factorization problem, the answers to both questions are affirmative [12].

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