Digital Signal Processing

Fast Fourier Transform (FFT)

- Complexity DFT
- Cooley-Tukey (CT) Decomposition
- Complexity CT
- Radix-2 FFT
We count real fixed-point or floating-point operations (additions and multiplications)

- memory access, indexing, loop counting, etc should not be ignored, but since they depend on specific architecture they are not counted
- for DSP microprocessors we have to compute the number of MACs (multiply/accumulate)

Complexity DFT

- \((N - 1)^2\) complex multiplications and \(N(N - 1)\) complex additions
- \(4(N - 1)^2\) real multiplications and \(2(N - 1)^2 + 2N(N - 1) = 4(N - 1)(N - 0.5)\) real additions
- \(W_{N}^{-kn}\) factors are generally computed off-line
Cooley-Tukey Decomposition

Reduce complexity by grouping same operations in \( X[k] = \sum_{n=0}^{N-1} x[n] W_N^{-kn} \)

- suppose we can factor \( N \) as \( N = PQ \)
- rewrite time index as \( n = Pq + p \) and frequency index as \( k = Qs + r \)
- \( W_N^{-kn} = W_N^{-(Qs+r)(Pq+p)} = W_N^{-Ns} W_N^{-rp} W_N^{-Qsp} W_N^{-Pqr} = W_N^{-rp} W_P^{-sp} W_Q^{-qr} \)
- DFT can then be rewritten as

\[
X[Qs + r] = \sum_{p=0}^{P-1} \sum_{q=0}^{Q-1} x[Pq + p] W_N^{-rp} W_P^{-sp} W_Q^{-qr}
\]

\[
= \sum_{p=0}^{P-1} W_N^{-rp} \left( \sum_{q=0}^{Q-1} x[Pq + p] W_Q^{-qr} \right) W_P^{-sp}
\]

\[
= \sum_{p=0}^{P-1} \left[ W_N^{-rp} X_P[r] \right] W_P^{-sp} = Y_r[s]
\]

- this continues until we only have prime-length DFTs
Cooley-Tukey Decomposition

\( N = 12, P = 3, Q = 4 \rightarrow \text{time-decimated CT} \)
Cooley-Tukey Decomposition

\(N = 12, P = 4, Q = 3 \rightarrow \text{frequency-decimated CT}\)
Cooley-Tukey Decomposition

- factors $\omega_{N}^{-rP}$ are called twiddle factors
- $P$ smallest prime: time-decimated FFT $\rightarrow$ decimation at time-domain side
- $Q$ smallest prime: frequency-decimated FFT $\rightarrow$ decimation at frequency-domain side
- different DFT sizes: mixed-radix FFT
- $N = P^r$ with $P$ prime: radix-$P$ FFT
- Inverse FFT (IFFT) is similar, except for
  - sign of exponent of twiddle factors
  - additional $1/N$ factor
Complexity CT Decomposition

- 1 DFT of size $N = PQ$ is replaced by $Q$ DFTs of size $P$ and $P$ DFTs of size $Q$ plus $PQ$ additional twiddle factors.

- Complexity is given by
  
  - $Q(P - 1)^2 + P(Q - 1)^2 + (P - 1)(Q - 1) = N(P + Q - 3) + 1$ complex multiplications
  
  - $QP(P - 1) + PQ(Q - 1) = N(P + Q - 2)$ complex additions

- Complexity advantage if $P + Q \ll N$

- Suppose $A_c(N)$ and $M_c(N)$ is the number of complex additions and multiplications for an $N$-point DFT, then we get
  
  - $M_c(N) = QM_c(P) + PM_c(Q) + (P - 1)(Q - 1)$
  
  - $A_c(N) = QA_c(P) + PA_c(Q)$

- Complexity of FFT can be computed by recursively applying these formulas.
CT decomposition with $P = 2$ and $Q = N/2$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{-kn} = \sum_{n=0}^{N/2-1} x[2n] W_N^{-2kn} + \sum_{n=0}^{N/2-1} x[2n + 1] W_N^{-(2n+1)}$$

$$= \sum_{n=0}^{N/2-1} x^e[n] W_N^{-kn} + W_N^{-k} \sum_{n=0}^{N/2-1} x^o[n] W_N^{-kn}$$

- $0 \leq k < N/2$:

$$X[k] = X^e[k] + W_N^{-k} X^o[k]$$

- $N/2 \leq k < N$: $k = N/2 + k'$ with $0 \leq k' < N/2$

$$X[k] = \sum_{n=0}^{N/2-1} x^e[n] W_N^{-(N/2+k')n} + W_N^{-(N/2+k')} \sum_{n=0}^{N/2-1} x^o[n] W_N^{-(N/2+k')n}$$

$$= \sum_{n=0}^{N/2-1} x^e[n] W_N^{-k'n} - W_N^{-k'} \sum_{n=0}^{N/2-1} x^o[n] W_N^{-k'n}$$

$$= X^e[k'] - W_N^{-k'} X^o[k']$$
Time-Decimated Radix-2 FFT


\[ W_{8^{-1}} \quad W_{8^{-2}} \quad W_{8^{-3}} \]
Frequency-Decimated Radix-2 FFT

CT decomposition with $P = N/2$ and $Q = 2$

\[ X[k] = \sum_{n=0}^{N-1} x[n]W_N^{-kn} = \sum_{n=0}^{N/2-1} x[n]W_N^{-kn} + \sum_{n'=0}^{N/2-1} x[N/2 + n']W_N^{-k(N/2+n')} \]

\[ = \sum_{n=0}^{N/2-1} x[n]W_N^{-kn} + W_2^{-k} \sum_{n'=0}^{N/2-1} x[N/2 + n']W_N^{-kn'} \]

- $k = 2p$ with $0 \leq p < N/2$:

\[ X[2p] = \sum_{n=0}^{N/2-1} x[n]W_N^{-pn} + \sum_{n'=0}^{N/2-1} x[N/2 + n']W_{N/2}^{-pn'} = \sum_{n=0}^{N/2-1} (x[n] + x[N/2 + n])W_{N/2}^{-pn} \]

- $k = 2p + 1$ with $0 \leq p < N/2$:

\[ X[2p + 1] = \sum_{n=0}^{N/2-1} x[n]W_N^{-(2p+1)n} - \sum_{n'=0}^{N/2-1} x[N/2 + n']W_N^{-(2p+1)n'} \]

\[ = \sum_{n=0}^{N/2-1} (x[n]W_N^{-n})W_{N/2}^{-pn} - \sum_{n'=0}^{N/2-1} (x[N/2 + n']W_N^{-n'})W_{N/2}^{-pn'} \]

\[ = \sum_{n=0}^{N/2-1} [(x[n] - x[N/2 + n])W_N^{-n}]W_{N/2}^{-pn} \]
Frequency-Decimated Radix-2 FFT

\[
\begin{align*}
x[0] & \\
x[1] & \\
x[2] & \\
x[3] & \\
x[4] & \\
x[5] & \\
x[6] & \\
x[7] & \\
\end{align*}
\]

\[
\begin{align*}
x[0] + x[1] & \\
\end{align*}
\]

\[
\begin{align*}
X[0] & \\
X[1] & \\
X[2] & \\
X[3] & \\
X[4] & \\
X[5] & \\
X[6] & \\
X[7] & \\
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

\[
\begin{align*}
W_8^{-1} & \\
W_8^{-2} & \\
W_8^{-3} & \\
\end{align*}
\]
Complexity Radix-2 FFT

- We define the complexity as the number of complex operations.

- We define $C_N$ as the complexity of an $N$-point DFT.

- The complexity can then be written as:

$$C_N = 2 \times C_{N/2} + \frac{3}{2}N$$

$$= 2 \times \left[ 2 \times C_{N/4} + \frac{3}{2} \frac{N}{2} \right] + \frac{3}{2}N$$

$$= 4 \times C_{N/4} + 2 \times \frac{3}{2}N$$

$$= x \times C_{N/x} + \log_2(x) \times \frac{3}{2}N$$

$$= N/2 \times C_2 + \log_2(N/2) \times \frac{3}{2}N$$

$$= \log_2(N) \times \frac{3}{2}N$$

- This should be compared with the roughly $2N^2$ complex operations if we adopt the standard DFT formula.