Non-parametric spectral estimation

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Goal: to evaluate the power spectral density of a discrete time, sampled stationary process and assess its quality

Literature: P. Stoica and R. Moses

*Introduction to spectral analysis*

Prentice Hall, 1997
Periodogram/correlogram

Object of interest: a stationary, zero mean stochastic process $y$ with samples $y[n]$

Definition of the power spectral density: $W_y(\theta) = \sum_{n=-\infty}^{\infty} r(n)e^{-jn\theta}$

with $r(n) = E(y^*[m]y[m+n])$

Candidates for PSD estimation:

the periodogram: $\Phi_y^p = \frac{1}{N} \left[ \sum_{n=0}^{N-1} y[n]e^{-jn\theta} \right]^2 = \frac{1}{N} |Y^d(\theta)|^2$

the correlogram: $\Phi_y^c(\theta) = \sum_{n=-(N-1)}^{N-1} \hat{r}[n]e^{-jn\theta}$

where $\hat{r}[n]$ is an estimate of the ‘covariance lag’ $r[n]$
Estimates of the covariance lag

Standard unbiased: \[ r^*[k] = \frac{1}{N-k} \sum_{n=k}^{N-1} y[n-k]y[n], \quad 0 \leq k \leq N - 1 \]

Standard biased: \[ r^*[k] = \frac{1}{N} \sum_{n=k}^{N-1} y[n-k]y[n], \quad 0 \leq k \leq N - 1 \]

(and complex conjugates for \( r[-k] \))
Property 1

If the standard biased estimates for the correlation lags are used, then

the correlogram = the periodogram

*the proof is based on expressing the periodogram as a double sum and re-sum along diagonals!*
Discussion

• Estimation errors on $r^k$ are of the order of $1/\sqrt{N}$ for large $N$ because it is the sum of a large sum of samples (law of large numbers) – at least for moderate values of $k$. But each value of the periodogram involves $2N-1$ such terms, hence its variance will not go to zero as $N \rightarrow \infty$.

• Problems can be expected if $r[k]$ goes to zero too slowly. The bias can easily be estimated:

$$E(r^k) = E\left\{\frac{1}{N} \sum_{n=k}^{N-1} y^*[n-k]y[n]\right\} = \frac{1}{N} (N - |k|)r[k]$$
Properties of the periodogram

Issues:  
* resolution in frequency  
* smearing (how much?)

\[
E[\Phi_y^p(\theta)] = \sum_{n=-N+1}^{N-1} E[r^\wedge(n)]e^{-jn\theta} \quad \text{with}
\]

\[
E(r^\wedge[n]) = (1 - \frac{|n|}{N})r[n]
\]

hence: \[
E(\Phi_y^p) = \sum_{n=-N+1}^{N-1} w_B[n]r[n]e^{-jn\theta},
\]

\[
w_B[n] = \left\{
\begin{array}{ll}
1 - \frac{|n|}{N}, & n = 0, \pm 1, \ldots, \pm(N-1) \\
0 & \text{elsewhere}
\end{array}
\right.
\]
Properties (2)

The covariance gets multiplied by the ‘Bartlett window’:

\[ w_B[n] \]

By the convolution theorem, with

\[ W_B^d(\theta) = \sum_{n=-N}^{N-1} \frac{N}{N} e^{-jn\theta} \]

\[ E(\Phi_y^p(\theta)) = \int_{-\pi}^{\pi} W_B^d(\theta - \psi) W_y(\psi) \frac{d\psi}{2\pi} \]
A summing gimmick

What is $W_B^d(\theta)$?

$$S_p = 1 + e^{-j\theta} + \cdots + e^{-j(N-1)\theta} = \frac{1 - e^{-jN\theta}}{1 - e^{-j\theta}}$$

$$N \ast W_B^d(\theta) = [1 + e^{j\theta} + \cdots + e^{j(N-1)\theta}]S_p = \left(\frac{1 - e^{jN\theta}}{1 - e^{j\theta}}\right)S_p$$
The Bartlett window

\[ w^d_B(\theta) = \frac{1}{N} \left| \frac{1 - e^{jN\theta}}{1 - e^{j\theta}} \right|^2 = \frac{1}{N} \left| \frac{\sin \left( \frac{N\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} \right|^2 \]

Resolution!
Smearing!
Unbiased estimator: Dirichlet window

Unbiased estimator: \[ E(r^\wedge(k)) = \frac{1}{N-k} \sum_{n=k}^{N-1} r(k) = r(k) \]

Rectangular window of size 2N-1:

\[ W_R(\theta) = \frac{\sin[(N - \frac{1}{2})\theta]}{\sin\frac{\theta}{2}} \]

- Not a positive window!
- Resolution more or less as before
- More smearing than before
The Bartlett window
Detail Bartlett (in real value)
Bartlett vs. Dirichlet

Figure 2.1. $W_B(\omega)/W_B(0)$, for $N = 25$.  

Figure 2.2. $W_R(\omega)/W_R(0)$, for $N = 25$.  

Other windows

• Law of large numbers: take averages over smaller sums (say of $M$ samples)
• Resolution decreases: order $1/M$ instead of $1/N$
• Smearing can be decreased substantially (Blackman-Tukey collection of windows)
• FFT hard to utilize with other windows.
Other windows (1)
Other windows (2)

Figure 2.4. The DTFTs of the window functions in Figure 2.3.