4. THE CONSTANT MODULUS ALGORITHM

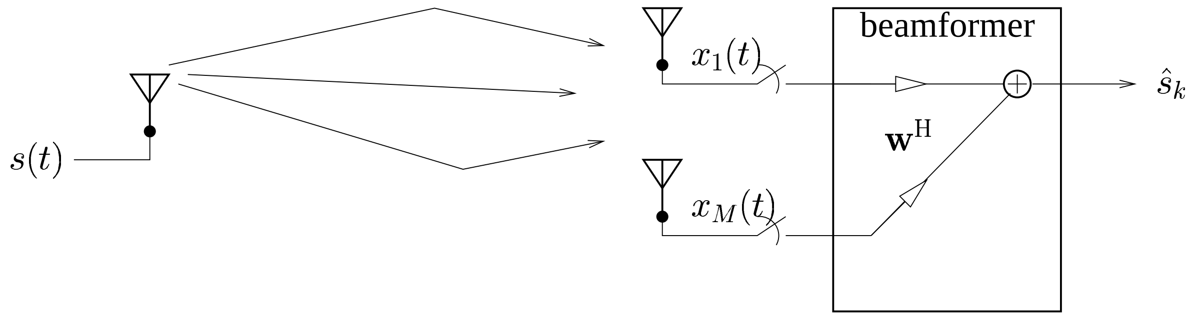
Outline

1. Introduction
2. CMA algorithm derivation
3. Simulations
4. Multistage CMA to find all sources

Introduction

- Many communication signals have the constant modulus (CM) property:
  FM, PM, FSK, PSK, ...
- If these are corrupted by noise/interference, the CM property is lost
- Can we find a filter **w** to restore this property, without knowing the sources?
- The answer is yes. It is obtained by the constant modulus algorithm (CMA)

Data model



- We receive 1 signal with noise (plus interference)

$$\mathbf{x}_k = \mathbf{a}s_k + \mathbf{n}_k$$

- The source is unknown but has *constant modulus*: $|s_k| = 1$ for all k.

Objective: construct a receiver weight vector **w** such that

$$y_k = \mathbf{w}^{\mathrm{H}}\mathbf{x}_k = \hat{s}_k$$

Possible solution: look for a **w** such that $|y_k| = 1$ for all k.

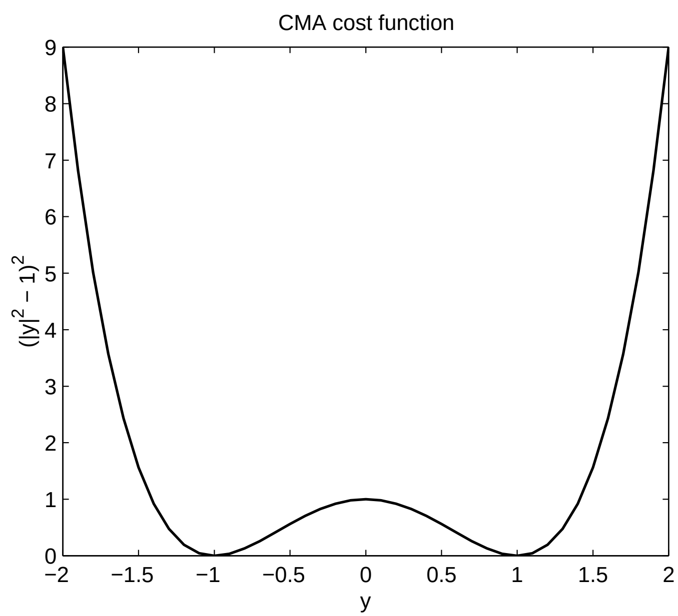
TU Delft

Cost function

- Possible optimization problem:

$$\min_{\mathbf{w}} J(\mathbf{w}) \quad \text{where} \quad J(\mathbf{w}) = \mathrm{E}\left[(|y_k|^2 - 1)^2\right]$$

- The CMA cost function as a function of y (for simplicity, y is taken real here):

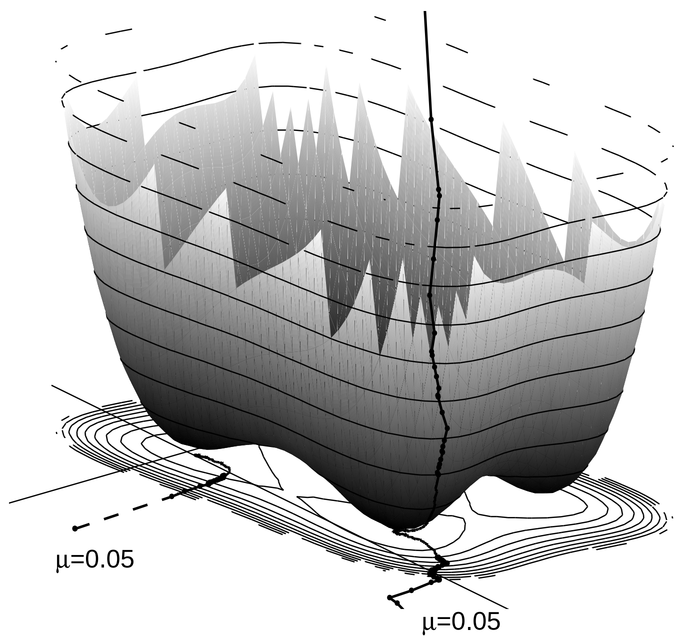


- There is no unique minimum:

  if $y_k = \mathbf{w}^{\mathrm{H}}\mathbf{x}_k$ is CM, then another beamformer is $\alpha\mathbf{w}$, for any scalar $|\alpha| = 1$

Cost function

- Cost function for 2 real sources and 2 antennas:



Constant modulus algorithm

- Cost function:

$$J(\mathbf{w}) = \mathrm{E}\Big[(|y_k|^2 - 1)^2\Big], \qquad y_k = \mathbf{w}^{\mathrm{H}}\mathbf{x}_k$$

- Stochastic gradient method:

$$\mathbf{w}_{k+1} = \mathbf{w}_k - \mu \nabla J(\mathbf{w}_k), \qquad \mu > 0 \text{ is the step size}$$

- Computation of gradient (use $|y_k|^2 = y_k \bar{y}_k = \mathbf{w}^{\mathrm{H}}\mathbf{x}_k \mathbf{x}_k^{\mathrm{H}}\mathbf{w}$):

$$\begin{aligned} \nabla J(\mathbf{w}) &= 2\mathrm{E}\{(|y_k|^2 - 1)\cdot \nabla(\mathbf{w}^{\mathrm{H}}\mathbf{x}_k\mathbf{x}_k^{\mathrm{H}}\mathbf{w})\} \\ &= 2\mathrm{E}\,\{(|y_k|^2 - 1)\cdot \mathbf{x}_k\mathbf{x}_k^{\mathrm{H}}\mathbf{w}\} \\ &= 2\mathrm{E}\,\{(|y_k|^2 - 1)\,\bar{y}_k\,\mathbf{x}_k\} \end{aligned}$$

- Replace expectation by instantaneous value and absorbe the factor 2 in μ:

$$\text{CMA(2,2):} \quad \begin{cases} y_k &= \mathbf{w}^{(k)\mathrm{H}}\mathbf{x}_k \\ \mathbf{w}^{(k+1)} &= \mathbf{w}^{(k)} - \mu\mathbf{x}_k\,(|y_k|^2 - 1)\bar{y}_k \end{cases}$$

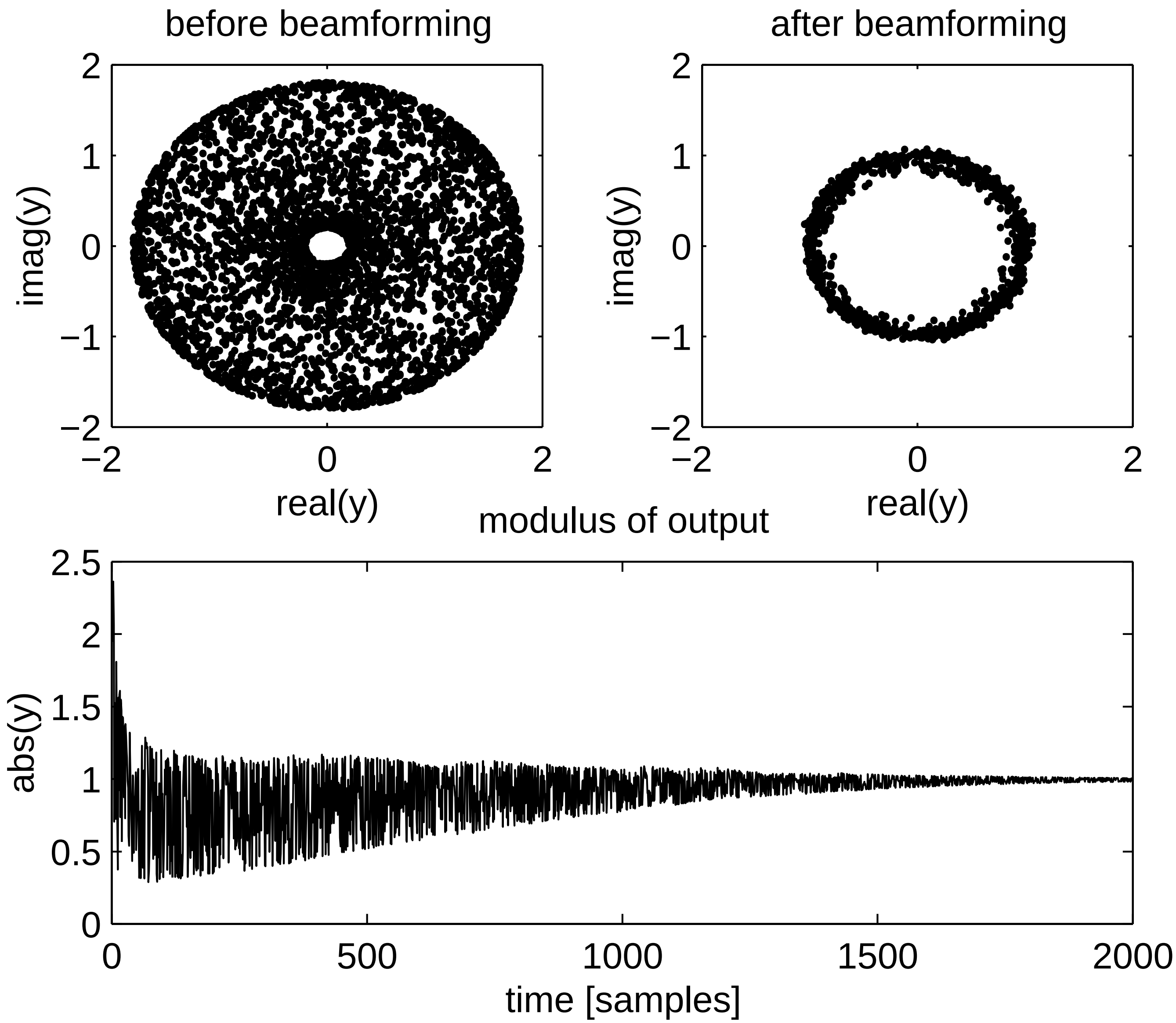
- Similar to LMS, but with update error $(|y_k|^2 - 1)y_k$.

Constant modulus algorithm

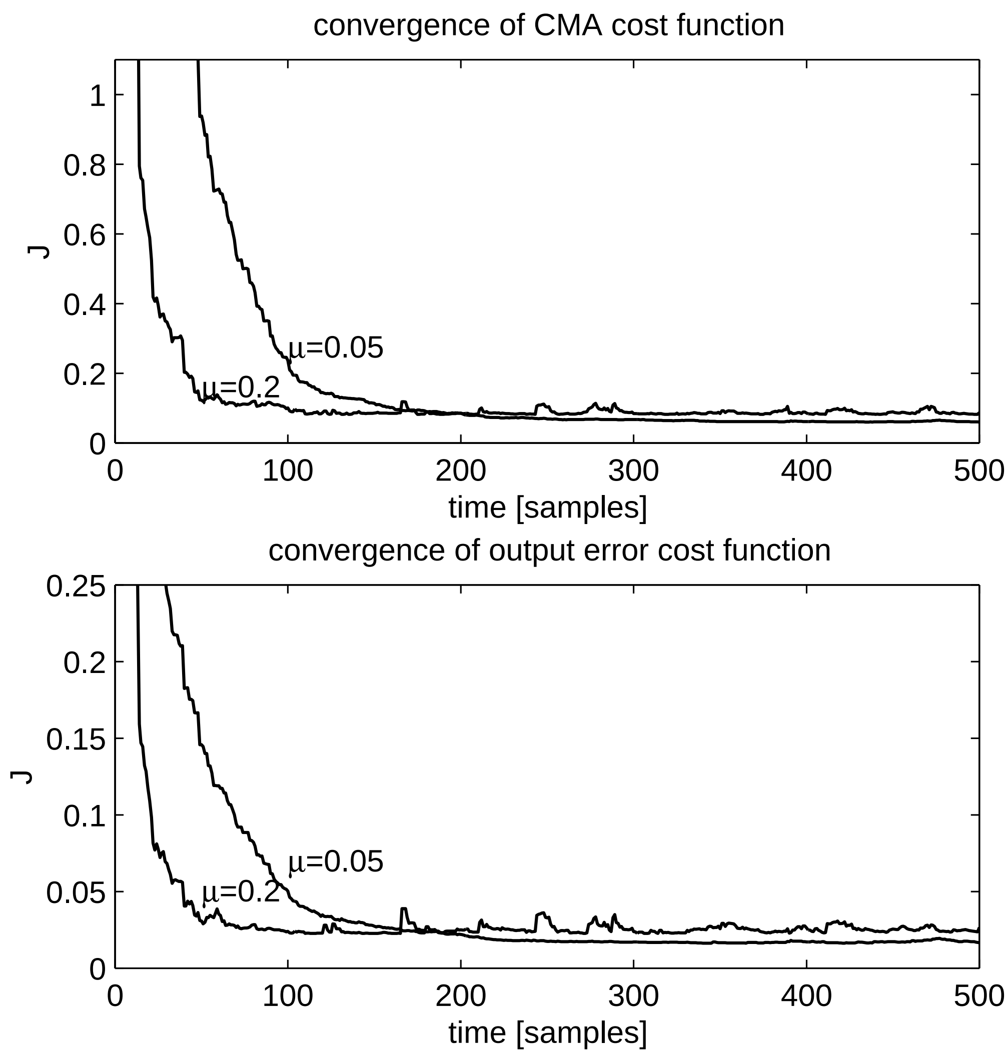
- Advantages
  - The algorithm is extremely simple to implement
  - Adaptive tracking of sources
  - Converges to minima close to the Wiener beamformers (for each source)

- Disadvantages
  - Noisy and slow
  - Step size μ should be small, else instable
  - Only one source is recovered (which one?)
  - Possible misconvergence to local minimum (with finite data)

Simulations

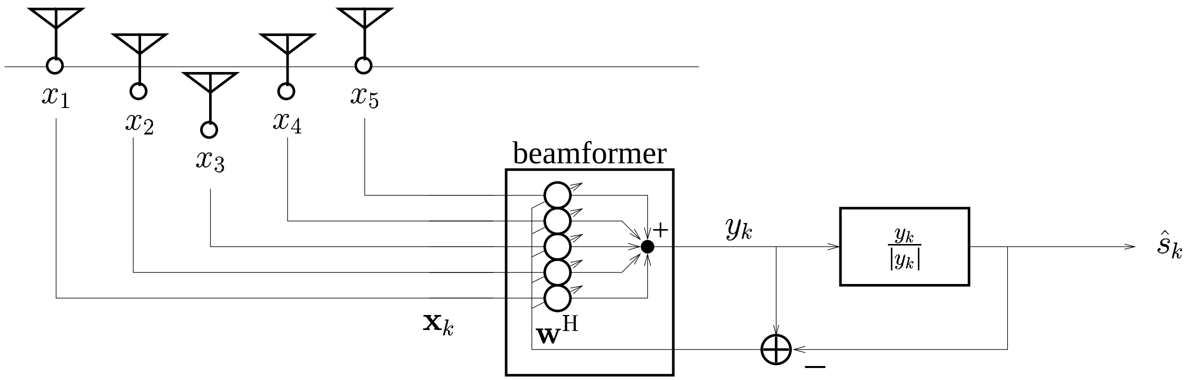


Simulations



As with LMS, a larger step size makes the convergence faster but also more noisy.

Other CMAs



- Alternative cost function: **CMA(1,2)**

$$J(\mathbf{w}) \;=\; \mathrm{E}(|y_k| - 1)^2 = \mathrm{E}(|\mathbf{w}^{\mathrm{H}}\mathbf{x}_k| - 1)^2$$

- Corresponding CMA iteration

$$\begin{cases} y_k & := \quad \mathbf{w}^{(k)H}\mathbf{x}_k \\ \mathbf{w}^{(k+1)} & := \quad \mathbf{w}^{(k)} - \mu\mathbf{x}_k(\bar{y}_k - \frac{\bar{y}_k}{|y_k|}) \end{cases}$$

- Similar to LMS, but with update error $y_k - \frac{y_k}{|y_k|}$.
- The desired signal is estimated by $\hat{s}_k = \frac{y_k}{|y_k|}$.

Other CMAs

- **Normalized CMA (NCMA)**: μ becomes scaling independent

$$\mathbf{w}^{(k+1)} := \mathbf{w}^{(k)} - \frac{\mu}{\|\mathbf{x}_k\|^2}\mathbf{x}_k\,(\bar{y}_k - \frac{\bar{y}_k}{|y_k|})$$

- **Orthogonal CMA (OCMA)**: whiten using data covariance **R**

$$\mathbf{w}^{(k+1)} := \mathbf{w}^{(k)} - \mu\mathbf{R}_k^{-1}\mathbf{x}_k\,(\bar{y}_k - \frac{\bar{y}_k}{|y_k|})$$

- **Least Squares CMA**: block update, we iteratively solve

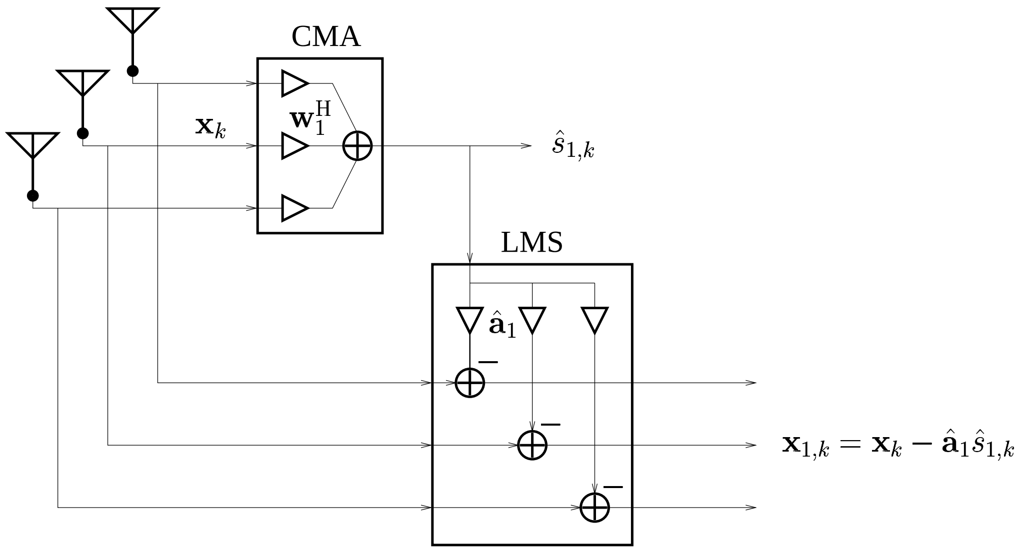
$$\min_{\mathbf{w}} \|\hat{\mathbf{s}} - \mathbf{w}^{\mathrm{H}}\mathbf{X}\|$$

where $\hat{\mathbf{s}}$ is the best blind estimate of the *complete* source vector based on **w**

$$\begin{cases} y_i = \mathbf{w}^{(k)H}\mathbf{x}_i \quad \text{for} \quad i = 1, 2, \ldots, N \\ \hat{\mathbf{s}}^{(k)} := [\frac{y_1}{|y_1|}, \frac{y_2}{|y_2|}, \cdots, \frac{y_N}{|y_N|}] \\ \mathbf{w}^{(k+1)} := (\hat{\mathbf{s}}^{(k)}\mathbf{X}^{\dagger})^{\mathrm{H}} \end{cases}$$

The CM Array

- Try to find all sources...



- CMA gives estimate of source 1: $\hat{s}_{1,k}$
- This is used as reference signal for LMS model matching:

$$\hat{\mathbf{a}}_1^{(k+1)} = \hat{\mathbf{a}}_1^{(k)} - \mu_{lms} [\hat{\mathbf{a}}_1^{(k)} \hat{s}_{1,k} - \mathbf{x}_k] \bar{\hat{s}}_{1,k}$$

- We can then remove this source and continue to find other sources:

$$\mathbf{x}_{1,k} = \mathbf{x}_k - \hat{\mathbf{a}}_1^{(k)} \hat{s}_{1,k}$$